Kenderdine Maths Tutoring
Solutions by Dr Richard Kenderdine

(1) From the Australian Mathematics Competition, Senior Division 2009:

The reciprocals of four positive integers add up to $\frac{19}{20}$.

Three of these integers are in the ratio 1 : 2 : 3. What is the sum of the four integers?

*(This problem is not especially difficult, what is important is obtaining a solution in the most efficient manner).*

Let the integers be a, 2a, 3a and b

$$\frac{1}{a}+\frac{1}{2a}+\frac{1}{3a}+\frac{1}{b}=\frac{19}{20}$$

$$\frac{11}{6a}+\frac{1}{b}=\frac{19}{20}$$

$$\frac{11b+6a}{6ab}=\frac{19}{20}$$

$$\frac{11b+6a}{3ab}=\frac{19}{10}$$

$$110b+60a=57ab$$

$$110b=57ab-60a$$
$$=(57b-60)a$$

$$\frac{110b}{57b-60}=a$$

$$\frac{110}{57-\frac{60}{b}}=a$$

Since $\frac{110}{55}=2$ then $b=30$ and $a=2$

Hence the integers are 2, 4, 6 and 30

Sum = 42

(1) For Extension 2 students who have studied integration, taken from the Cambridge text:

Find $\displaystyle\int_0^a \sqrt{\frac{1+u}{1-u}}\,du$

*(Which method is preferred – substitution, parts or something else?).*

Something else to start with

$$\sqrt{\frac{1+u}{1-u}} = \sqrt{\frac{1+u}{1-u} \times \frac{1+u}{1+u}} = \frac{1+u}{\sqrt{1-u^2}}$$

$$\therefore \int_0^a \sqrt{\frac{1+u}{1-u}}\,du = \int_0^a \frac{1+u}{\sqrt{1-u^2}}\,du = \int_0^a \frac{1}{\sqrt{1-u^2}}(1+u)\,du$$

By parts we have

$$\left[\sin^{-1} u\right](1+u)\Big|_0^a - \int_0^a (\sin^{-1} u) \times 1\,du$$

$$= (\sin^{-1} a)(1+a) - 0 - \left[u \sin^{-1} u\Big|_0^a - \int_0^a \frac{u}{\sqrt{1-u^2}}\,du\right]$$

$$= (1+a)\sin^{-1} a - a\sin^{-1} a + \left(-\sqrt{1-u^2}\right)\Big|_0^a$$

$$= \sin^{-1} a - \left[\sqrt{1-a^2} - 1\right]$$

$$= \sin^{-1} a + 1 - \sqrt{1-a^2}$$

(with the restriction $a \leq 1$)