Slope Fields for First-order Differential Equations

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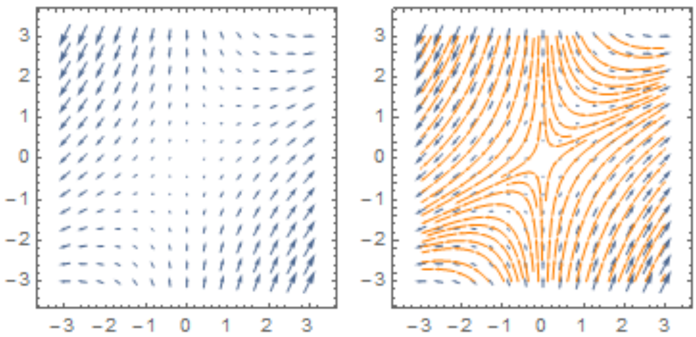
This note provides details for sketching and interpreting slope fields for first-order differential equations when expressed in the form $y' = f(x, y)$. Examples are taken from the Cambridge Year 12 Extension 1 textbook.

The text has diagrams of slope fields that consist of short intervals showing the gradient at various points. *Mathematica* plots a vector field that is similar, shown on the left below, and also provides a streamplot that shows a selection of solutions to the differential equation (DE). Integration constants exist in solutions to DEs and we need to know the coordinates of a point to find the particular solution. If all the possible solutions were shown on a plot then the plot would just be a filled region as the constant is continuous. The streamplot applicable to the vector plot is shown on the right:



Isoclines are curves that join points on the streamlines (solutions) for which the gradients of the tangents are constant.
In each of the examples the solution of the DE is provided and the streamplot is shown together with at least one solution and isocline.

Example 1: $y' + y = x$

Consider the differential equation $y' + y = x$ so that $y' = x - y$. To sketch the slope field note that:

(1) $y' = 0$ along the line $y = x$
(2) for fixed $x > y + 1$ we have $y' > 0$ and decreasing as y increases
(3) for fixed $x < y$ we have $y' < 0$ and $\longrightarrow -\infty$ as $y \longrightarrow \infty$

To solve $y' + y = x$ we need to find two solutions, one to $y' + y = 0$ and the other to $y' + y = x$. This is because the first solution effectively hides in the background but obviously effects the solution.

We solve $y' + y = 0$ by writing $y' = -y$ ie $\frac{1}{y}\frac{dy}{dx} = -1$

$$\text{so } \int \frac{1}{y}\, dy = \int -1\, dx \Longrightarrow \ln(y) = -x + c \Longrightarrow y = K e^{-x}$$

To solve $y' + y = x$ we let $y = A x + B \Longrightarrow y' = A$

$$\text{Then } y' + y = A + A x + B \Longrightarrow A = 1 \text{ and } B = -1$$

So the full solution is $\boldsymbol{y = K e^{-x} + x - 1}$

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Show[StreamPlot[{1, x - y}, {x, -3, 3}, {y, -3, 3}],
 Plot[e^-x + x - 1, {x, -3, 3}, PlotStyle → {Blue, Thick}],
 Plot[-2 e^-x + x - 1, {x, -3, 3}, PlotStyle → {Green, Thick}],
 Plot[x - 1, {x, -3, 3}, PlotStyle → {Red, Thick}],
 Plot[x, {x, -3, 3}, PlotStyle → {Orange, Dashed, Thick}]]
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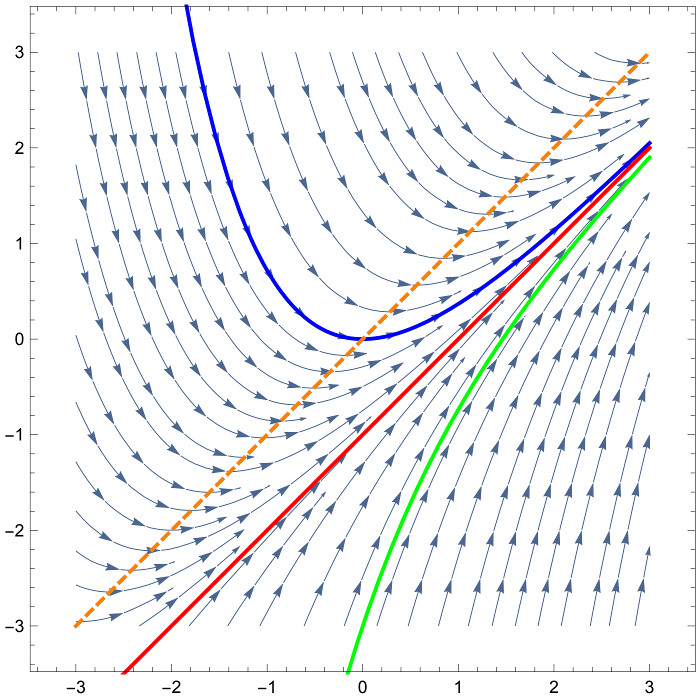

The plot shows the streamlines together with the line $y = x$ (where $y' = 0$), the line $y = x - 1$ (the particular solution) and two full solutions, one for $K = 1$ (blue) and the other for $K = -2$ (green). Note that $y = x - 1$ is an asymptote as the influence of the $K\,e^{-x}$ term in the solution decays quickly for increasing positive x.