Finding an inflection point for a rational function

Dr Richard Kenderdine

Kenderdine Maths Tutoring

Suppose we have to sketch the function $f(x) = \frac{x^2}{x^2+2x-3}$ and need to locate any inflection points. Usually the second derivative would be found by using the quotient rule twice. This is rather messy and results in having to solve a cubic equation.However by doing some preliminary work we can make the job easier in the end.

The first thing to do is to separate the function into simpler fractions. This approach is called partial fractions.

Either use polynomial division or, simpler, we can add terms to make the numerator the same as the denominator and then subtract the extra terms to balance:

$$f(x) = \frac{x^2}{x^2+2x-3} = \frac{x^2+2x-3-2x+3}{x^2+2x-3} = \frac{x^2+2x-3}{x^2+2x-3} + \frac{-2x+3}{x^2+2x-3}$$

$$= 1 + \frac{-2x+3}{x^2+2x-3}$$

$$= 1 + \frac{-2x+3}{(x-1)(x+3)}$$

Now we need to find two numbers A and B such that

$$\frac{A}{x-1} + \frac{B}{x+3} = \frac{-2x+3}{(x-1)(x+3)} \qquad (1)$$

One way to do this is combine the two fractions on LHS and solve two simultaneous equations. A simpler alternative is to multiply both sides by (x-1), cancel and substitute x = 1, giving $A = \frac{1}{4}$. Repeat the process by multiplying both sides of (1) by (x+3), cancelling and substituting x = -3 to give $B = -\frac{9}{4}$.

Hence we have $$f(x) = 1 + \frac{1}{4}\left(\frac{1}{x-1} - \frac{9}{x+3}\right) \qquad (2)$$

This is a combination of two hyperbola lifted up 1 unit (hence the horizontal asymptote is y = 1).

The first derivative is $$f'(x) = \frac{1}{4}\left(-\frac{1}{(x-1)^2} + \frac{9}{(x+3)^2}\right)$$

and the second derivative is $$f''(x) = \frac{1}{2}\left(\frac{1}{(x-1)^3} - \frac{9}{(x+3)^3}\right)$$

To find inflection points we need to solve $f''(x) = 0$.

We can treat $\frac{1}{(x-1)^3} - \frac{9}{(x+3)^3}$ as a difference of two cubes.

The factorisation of $a^3 - b^3$ is $(a - b)\left(a^2 + a\,b + b^2\right)$ and the only real number solution of $a^3 - b^3 = 0$ is $a = b$.

Hence to solve $\frac{1}{(x-1)^3} - \frac{9}{(x+3)^3} = 0$ we have $\frac{1}{x-1} - \frac{\sqrt[3]{9}}{x+3} = 0$ so

$$(x + 3) = \sqrt[3]{9}\,(x - 1)$$

giving $$3 + \sqrt[3]{9} = x\left(\sqrt[3]{9} - 1\right)$$

so $$x = \frac{3+\sqrt[3]{9}}{\sqrt[3]{9} - 1} \approx 4.703 \quad \text{(3 dp)}$$

Check this result with *Mathematica* (the notation $\partial_{x,x} f$ means differentiate f twice with respect to x ie the second derivative):

```
In[101]:= f = x^2/(x^2 + 2 x - 3);
          Solve[∂x,x f == 0, x] // N
Out[102]= {{x → 4.70342}, {x → -0.101708 - 0.972834 i}, {x → -0.101708 + 0.972834 i}}
```

There is one real solution (4.70342) and two complex solutions - these have no influence on the curve. Check for change of sign in the second derivative around the point:

```
In[148]:= ∂x,x f /. x → 4.7
Out[148]= 0.0000141839
```

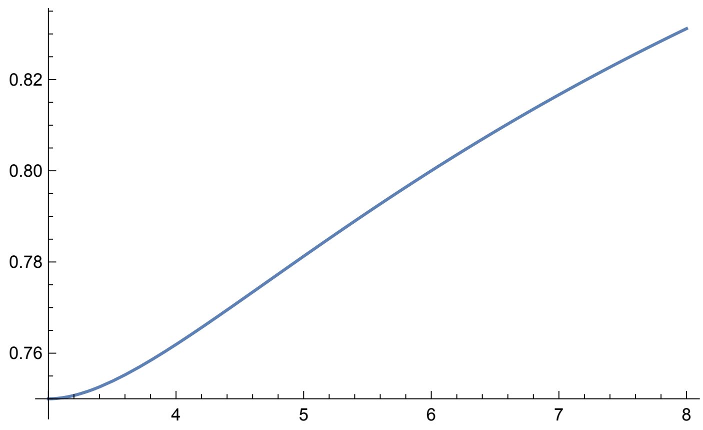
```
In[149]:= ∂x,x f /. x → 4.8
Out[149]= -0.000370512
```

The second derivative is positive when $x = 4.7$ and negative at $x = 4.8$, proving that $x = 4.70342$ is indeed an inflection point.

Here is a plot of the function in the domain (3, 8), the inflection appears to be around $x = 4.7$.

```
In[110]:= Plot[f, {x, 3, 8}]
```

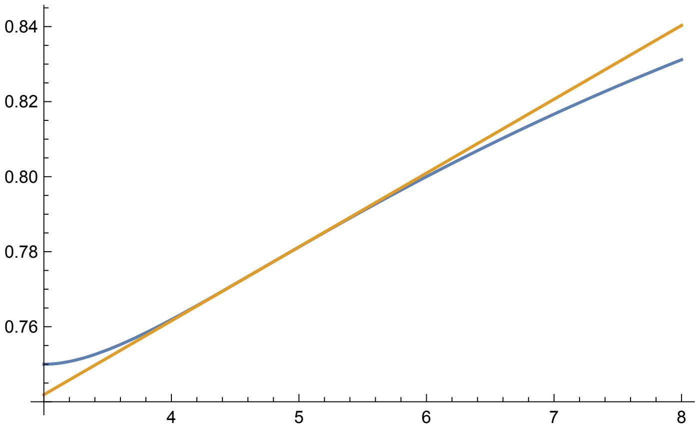
Out[110]=



The equation of the tangent at $x = 4.70342$ is $y = 0.0196876(x - 4.70342) + 0.775419$ and this is plotted with the function in the domain (3, 8):

In[150]:=
```
tl = 0.0196876 (x - 4.70342) + 0.775419;
Plot[{f, tl}, {x, 3, 8}]
```

Out[151]=



Finally, here is a plot of the function in the domain (-6, 10):

In[112]:=
```
Plot[f, {x, -6, 10}]
```

Out[112]=