A cubic inequality

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We are required to prove that, for *x, y* and *z* all positive,

$$(x+y+z)^3 \le 9\left(x^3+y^3+z^3\right)$$

To start, note that

$$(x+y+z)^3 = x^3+y^3+z^3+3\left(x^2\,y+x^2\,z+x\,y^2+z\,y^2+x\,z^2+y\,z^2\right)+6\,x\,y\,z$$

Now we use the factorisation of the sum of two cubes:

$$\begin{aligned} x^3+y^3 &= (x+y)\left(x^2-x\,y+y^2\right) \\ &= (x+y)\left(x^2-2\,x\,y+y^2+\,x\,y\right) \\ &= (x+y)\left((x-y)^2+\,x\,y\right) \\ &\ge (x+y)\,(x\,y) \\ &= x^2\,y+x\,y^2 \end{aligned}$$

Then also $\quad x^3+z^3 \ge x^2\,z+x\,z^2$

and $\quad y^3+z^3 \ge y^2\,z+y\,z^2$

giving $\quad x^2\,y+x^2\,z+x\,y^2+z\,y^2+x\,z^2+y\,z^2 \le\; 2\left(x^3+y^3+z^3\right)$

Now we have to look at the *xyz* term.

We start with

$$x^3+y^3+z^3-3\,x\,y\,z \;=\; (x+y+z)\left(x^2+y^2+z^2-x\,y\;-\;x\,z-y\,z\right)$$

Now $\quad (x-y)^2 \ge 0 \quad$ so $\quad x^2+y^2 \ge 2\,x\,y$

and similarly $\quad x^2+z^2 \ge 2\,x\,z \qquad y^2+z^2 \ge 2\,y\,z$

Adding these three gives

$$2\left(x^2+y^2+z^3\right) \;\ge\; 2\,(x\,y\;+\;x\,z+y\,z)$$

or $\quad x^2+y^2+z^3 \;\ge\; x\,y\;+\;x\,z+y\,z$

so $\quad x^2+y^2+z^3\;-\;x\,y\;-\;x\,z-y\,z \ge 0$

and since *x, y* and *z* are all positive,

$$x^3 + y^3 + z^3 - 3\,x\,y\,z \geq 0$$

so $$3\,xyz \leq x^3 + y^3 + z^3$$

or $$6\,xyz \leq 2(x^3 + y^3 + z^3)$$

Finally $$x^3 + y^3 + z^3 + 3\left(x^2\,y + x^2\,z + x\,y^2 + z\,y^2 + x\,z^2 + y\,z^2\right) + 6\,x\,y\,z$$

$$\leq x^3 + y^3 + z^3 + 3\left(2\left(x^3 + y^3 + z^3\right)\right) + 2\left(x^3 + y^3 + z^3\right)$$

$$= 9\left(x^3 + y^3 + z^3\right)$$

Giving the result

$$(x + y + z)^3 \leq 9\left(x^3 + y^3 + z^3\right)$$