Numbers on nodes and arcs

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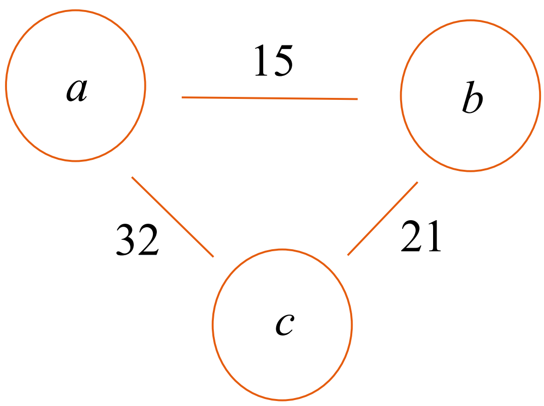
Puzzles exist that require finding the numbers at the nodes of a loop such that the numbers at adjacent nodes sum to the values on the connecting arc.

The simplest case to consider has three nodes. Consider the following example:



We have the equations:

$$a + b = 15 \qquad b + c = 21 \qquad a + c = 32$$

If we add these three equations together we have $2(a + b + c) = 68$ so $a + b + c = 34$.

Since $b + c = 21$ then $a = 13$ and then $b = 2$ and $c = 19$.

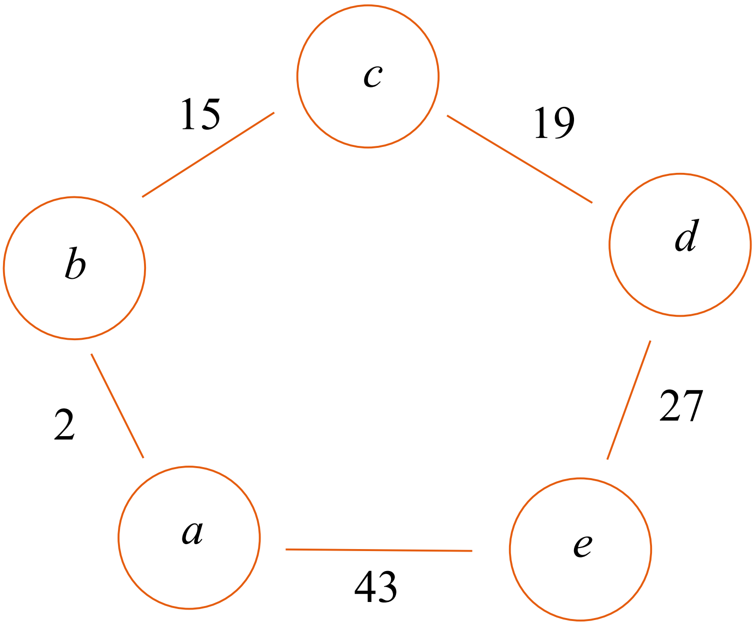
The method therefore is:

The unknown number at a node is found by subtracting the number on the opposite arc from half the sum of the numbers on the arcs.

Loops with an odd number of nodes

Now consider a loop with five nodes. The method used for three nodes will not work here. Therefore we need to find a method that will work for a loop with an odd number of nodes.

In the following diagram we want to find values for a - e such that $a + b = 28$, $b + c = 15$ etc.



One way to solve the problem is to set up a series of simultaneous equations. When we do this we find there is an elegant way to find the unknowns.

The basic equations are:

$$a + b = 28 \quad b + c = 15 \quad c + d = 19 \quad d + e = 27 \quad a + e = 43$$

We can use these to obtain the node values in terms of the starting node value, a:

$$b = 28 - a$$

$$c = 15 - b = 15 - (28 - a) = -13 + a$$

$$d = 19 - c = 19 - (-13 + a) = 32 - a$$

$$e = 27 - d = 27 - (32 - a) = - 5 + a$$

$$a = 43 - e = 43 - (-5 + a) = 48 - a$$

From the last equation we have $2a = 48$ so $a = 24$, giving $b = 4$, $c = 11$, $d = 8$, $e = 19$.

Now the interesting thing is if we didn't simplify each equation we would notice a pattern:

$$a = 43 - (27 - (19 - (15 - (28 - a)))) = 43 - 27 + 19 - 15 + 28 - a$$

So $2a = 43 - 27 + 19 - 15 + 28$

or $a = 48 / 2 = 24$

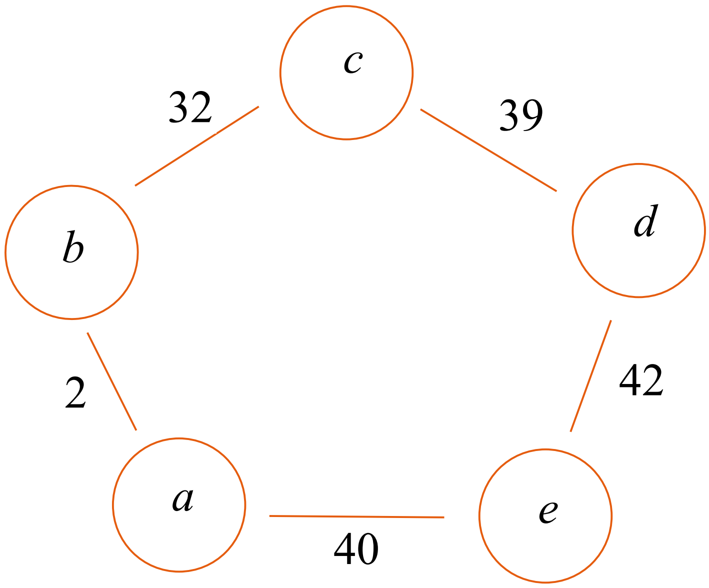
Equivalently we could have

$$a = \frac{1}{2}(28 - 15 + 19 - 27 + 43) = 24$$

This illustrates the method :

(1) *Start at any node and proceed round the loop in either direction*
(2) *Alternate the signs of the values on the arcs*
(3) *The number in the starting node is half the sum of the alternating arc values*
(4) *Find the remaining node values using the starting node and arc values*

Now try one for yourself:

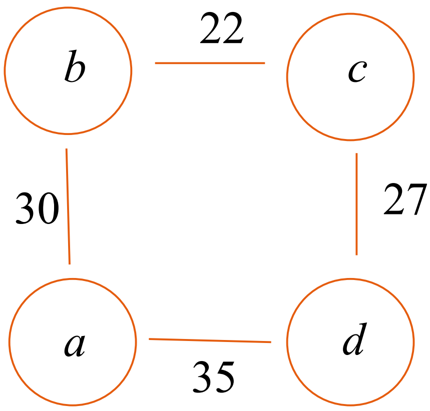


Loops with an even number of nodes

The above examples have 5 nodes. Will the method work for any number of nodes? It will for an odd number of nodes but consider what happens with an even number of nodes.

When we expressed each node value in terms of a the sign of a alternated between +ve and -ve. When there are an even number of nodes the final equation will be of the form $a = n + a$ which cannot be solved.

What does this mean for the solution? Let' s look at an example :



We have $a + b = 30$ $b + c = 22$ $c + d = 27$ $a + d = 35$

Thus $b = 30 - a$

$c = 22 - b = 22 - (30 - a) = -8 + a$

$d = 27 - c = 27 - (-8 + a) = 35 - a$

$a = 35 - d = 35 - (35 - a) = a$

The last equation cannot be solved. *The result of this is that there is not a unique solution.*

Here are three possible solutions for (*a, b, c, d*), assuming all values are positive: (10, 20, 2, 25), (15, 15, 7, 20), (8, 22, 0, 27).

This situation is equivalent to trying to solve two simultaneous equations in two unknowns where one equation is a multiple of the other. If we consider the equations as representing lines then here we have two coincident lines, thus giving an infinite number of solutions.

To solve a loop with an even number of nodes you can select any number for the first node then calculate the remaining nodes according to the values on the arcs.

Matrix methods

These problems can be solved by using matrix methods to set up and solve the simultaneous equations. If the matrix A contains the coefficients of the unknowns in the equations (either 0 or 1), the vector x the unknown node values and the vector b the numbers on the arcs then we solve $Ax = b$ from $x = A^{-1} b$, providing A has an inverse.

For example, the problem with five nodes would have

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}; \qquad x = \begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix}; \qquad b = \begin{pmatrix} 28 \\ 15 \\ 19 \\ 27 \\ 43 \end{pmatrix}$$

The solution, which agrees with the previous one, is obtained from

```
Inverse[A].b // Flatten
```

```
{24, 4, 11, 8, 19}
```

The corresponding inputs for the problem with four nodes are:

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}; \qquad x = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}; \qquad b = \begin{pmatrix} 30 \\ 22 \\ 32 \\ 40 \end{pmatrix}$$

If we try to solve this we obtain an error message saying the matrix A is singular ie it does not have an inverse and the input is returned:

```
Inverse[A].b // Flatten
```

```
Inverse[{{1, 1, 0, 0}, {0, 1, 1, 0}, {0, 0, 1, 1}, {1, 0, 0, 1}}].
 {{30}, {22}, {32}, {40}}
```

This tells us that the problem cannot be solved by matrices. It does not mean there is no solution for as we have seen there are many solutions.

Acknowledgement

This note was inspired by a student who wanted to see how long it took me to sove a similar puzzle. The student was subsequently shown how algebra can eficiently solve such puzzles and further-more provide insights into developing a suitable algorithm for doing so in all cases.