Distances from a point to complex roots of unity

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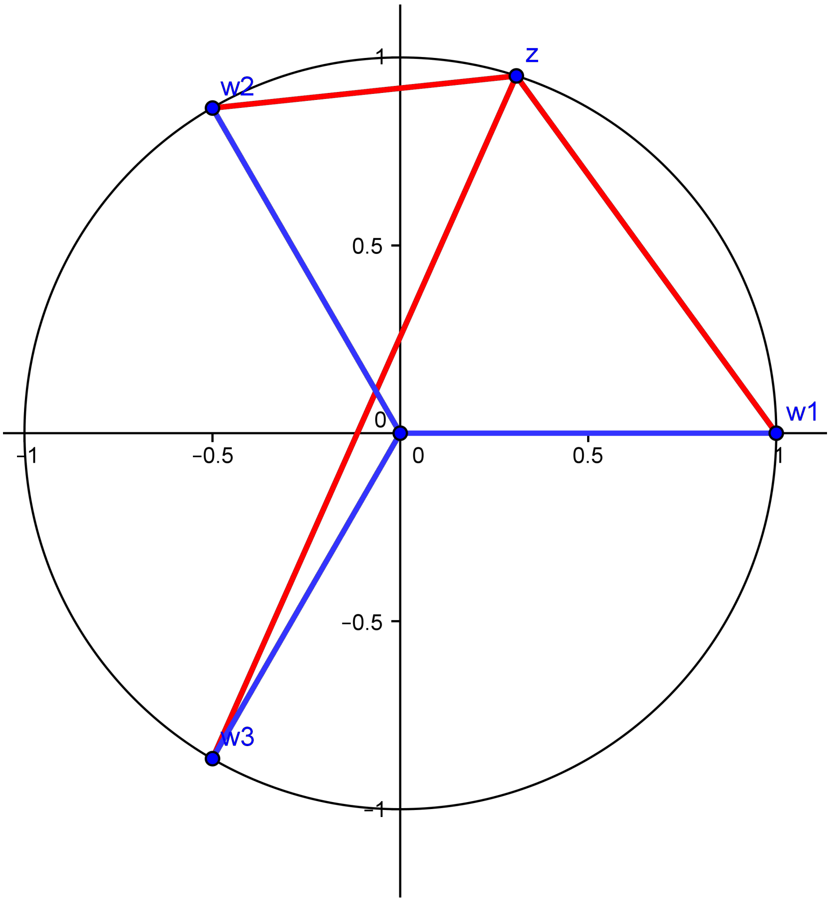
Problem:

We want to show that the sum of the squares of the Euclidean distances from any point on the unit circle in the Argand diagram to all of the n^{th} roots of unity equals $2n$ (for $n > 1$).

Example when $n = 3$:

The solutions of $z^3 = 1$ are 1, $\frac{-1+\sqrt{3}\,i}{2}$ and $\frac{-1-\sqrt{3}\,i}{2}$. These are shown as w_1, w_2 and w_3 in the plot below. We can consider a variable point z on the unit circle and we want to calculate the distances to w_1, w_2 and w_3 (shown in red).



As an example let z be the complex number with argument $\frac{2\pi}{5}$. We then calculate the sum of the square of the Euclidean distances to w_1, w_2 and w_3.

$$z = \left\{\text{Cos}\left[\frac{2\pi}{5}\right], \text{Sin}\left[\frac{2\pi}{5}\right]\right\} \text{ // N;}$$

$$\text{cuberoots} = \left\{\{1, 0\}, \left\{\text{Cos}\left[\frac{2\pi}{3}\right], \text{Sin}\left[\frac{2\pi}{3}\right]\right\}, \left\{\text{Cos}\left[\frac{4\pi}{3}\right], \text{Sin}\left[\frac{4\pi}{3}\right]\right\}\right\};$$

$$\sum_{k=1}^{3} (\text{EuclideanDistance}[z, \text{cuberoots}[[k]]])^2$$

6.

Thus the sum of the square of the distances is 6, which is twice the number of roots. We can prove this is the case for all values of $n > 1$.

Proof

Let P be the general point z on the unit circle with $z = x + i\,y$ and A_k the position of the root w_k with $w_k = a + i\,b$.

Then $P\,A_k{}^2 = (x - a)^2 + (y - b)^2$ as usual.

However $(z - w_k)\,(\overline{z} - \overline{w}_k) = (x + i\,y - (a + i\,b))(x - i\,y - (a - i\,b))$

$= (x - a + i(y - b))(x - a - i(y - b))$

$= (x - a)^2 + (y - b)^2$

So $PA_k{}^2 = (z - w_k)\,(\overline{z} - \overline{w}_k)$

Then $\sum_{k=1}^{n} P\,A_k{}^2 = \sum_{k=1}^{n} (z - w_k)\,(\overline{z} - \overline{w}_k)$

$= \sum_{k=1}^{n} (z\,\overline{z} - z\,\overline{w}_k - w_k\,\overline{z} + w_k\,\overline{w}_k)$

$= \sum_{k=1}^{n} z\,\overline{z} - z\sum_{k=1}^{n} \overline{w}_k - \overline{z}\sum_{k=1}^{n} w_k + \sum_{k=1}^{n} w_k\,\overline{w}_k$

Now on the unit circle $z\,\overline{z} = 1$ and $w_k\,\overline{w}_k = 1$ while $\sum_{k=1}^{n} w_k = \sum_{k=1}^{n} \overline{w}_k = 0$

Hence $\sum_{k=1}^{n} P\,A_k{}^2 = \sum_{k=1}^{n} 1 + \sum_{k=1}^{n} 1 = 2n$

A further question

Given a point P on the x-axis ($-1<x<1$), prove that the product of the Euclidean distances to all the n^{th} roots of unity equals $1 - x^n$

ie $\prod_{k=1}^{n} P\,A_k = 1 - x^n$

We have from above $PA_k = \sqrt{(x - w_k)\,(x - \overline{w}_k)}$ since $\overline{x} = x$

From the example of the cube roots of unity we see that $\overline{w}_1 = w_1$, $\overline{w}_2 = w_3$ and $\overline{w}_3 = w_2$.

Therefore we have in this case $\prod_{k=1}^{3} P\,A_k = \sqrt{(x - w_1)\,(x - \overline{w}_1)\,(x - w_2)\,(x - \overline{w}_2)\,(x - w_3)\,(x - \overline{w}_3)}$

$= \sqrt{(x - w_1)\,(x - w_1)\,(x - w_2)\,(x - w_3)\,(x - w_3)\,(x - w_2)}$

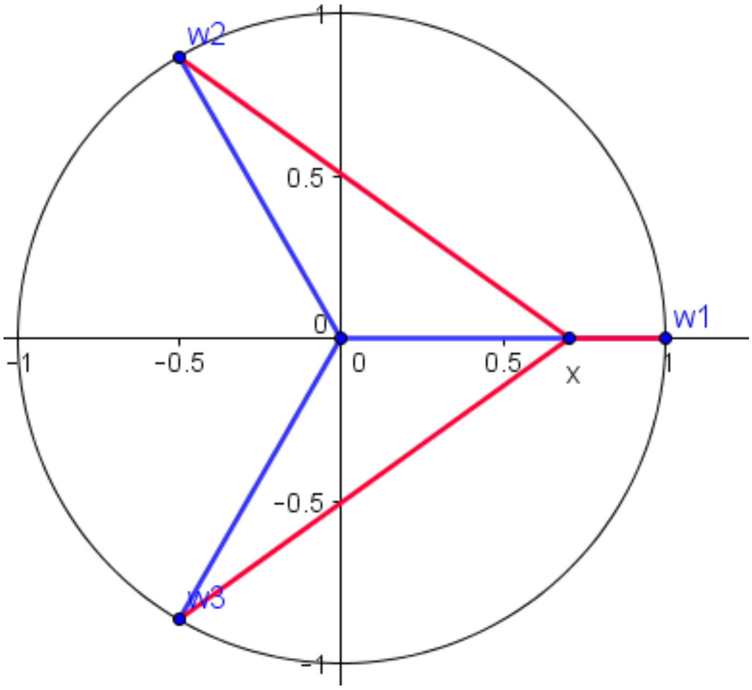
$= \sqrt{(x - w_1)^2\,(x - w_2)^2\,(x - w_3)^2}$

$= \pm(x - w_1)\,(x - w_2)\,(x - w_3)$

This is the factorisation of $\pm(x^3 - 1)$.

To determine which sign is appropriate we first need to consider the product $(x - w)(x - \overline{w})$ with $w = a + i\,b$ and $\overline{w} = a - i\,b$. We have $(x - w)(x - \overline{w}) = (x - a)^2 + b^2$ ie positive. Hence the sign of the product of the roots depends upon the sign of $(x - w_1)$ and since $w_1 = 1$ and $-1 < x < 1$ we have $(x - w_1) < 0$. Distances are positive and therefore we need to use $(w_1 - x)$, giving

$$\prod_{k=1}^{3} P\,A_k = (w_1 - x)\ (x - w_2)(x - w_3) = 1 - x^3$$



The extension to the n^{th} roots of unity is immediate since every root has a conjugate. The derivation of $\prod_{k=1}^{n} P\,A_k$ follows the same argument, yielding

$$\prod_{k=1}^{n} P\,A_k = (w_1 - x)\ (x - w_2)(x - w_3) = 1 - x^n$$