Summing a sequence with a pattern of ones

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Suppose we have the sequence 1,2,1,2,2,1,2,2,2,1,2,2,2,2,1,2,2,2,2,2,.....ie starting with single examples of 1 and 2 every further instance of 1 increases the number of following 2s by one.

We want to sum the first k terms of this sequence. One method is to split the sequence into blocks with each block commencing with 1 eg (1,2), (1,2,2), (1,2,2,2) etc. The total of each block creates an arithmetic series so we can easily find the sum of the block totals. To do this it is necessary to find the number of complete blocks in k terms and the remainder of the incomplete block.

A second method is to note that the 1s appear in positions 1, 3, 6,10, etc. These numbers are termed *triangular numbers* and obtained by the partial sums of the positive integers 1, 2, 3, 4, 5, etc (1, 1 + 2 = 3, 1 + 2 + 3 = 6 etc) and the n^{th} term is calculated from $\frac{1}{2}n(n+1)$ for $n \geq 1$.
All we have to do is calculate the value of n (denoted by n^*) that gives the highest triangular number that is less than or equal to k.The sum of the series will then be $2k - n^*$ since we have to reduce $2k$ by the number of 1s.

The value of n^* is the integer part of the positive solution of $\frac{1}{2}n(n+1) = k$

Let’s look at an example with $k = 2000$.

In[9]:=
```
k = 2000;
Solve[n (n + 1)/2 == k];
n /. %;
numones = Floor[%]
```

Out[12]=
```
{-64, 62}
```

There are 62 triangular numbers less than 2000 (the 62 nd triangular number is $\frac{1}{2}(62)(63) = 1953$) and therefore there are 62 ones in the sequence. The sum of the sequence is therefore:

In[13]:=
```
sumseries = 2 k - numones[[2]]
```

Out[13]=
```
3938
```

We can check this by calculating the positions of the 1s in the sequence, creating the sequence and finding the total.

In[15]:=
```
onespositions = Table[i (i + 1)/2, {i, 1, numones[[2]]}]
```

Out[15]=
```
{1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, 105, 120, 136, 153, 171, 190, 210,
 231, 253, 276, 300, 325, 351, 378, 406, 435, 465, 496, 528, 561, 595, 630,
 666, 703, 741, 780, 820, 861, 903, 946, 990, 1035, 1081, 1128, 1176, 1225,
 1275, 1326, 1378, 1431, 1485, 1540, 1596, 1653, 1711, 1770, 1830, 1891, 1953}
```

In[16]:=
```
twoseries = Table[If[MemberQ[onespositions, j] == True, 1, 2], {j, k}];
```

The first and last 17 terms in the sequence are shown:

In[17]:=
```
Short[twoseries]
```

Out[17]//Short=
```
{1, 2, 1, 2, 2, 1, 2, 2, 2, 1, 2, 2, 2, 2, 1, 2, 2,
 <<1966>>, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2}
```

Here is the total of the sequence:

In[18]:=
```
Total[twoseries]
```

Out[18]=
```
3938
```