2018 Extension 2 Mathematics
Question 16

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(a) Step 1: Prove true for $n = 1$

LHS $\quad x^3 - 1 = (x-1)\left(x^2 + x + 1\right)$ on factorising

RHS $\quad (x-1)\left(x^2 + x + 1\right)$

Therefore true for $n = 1$

Step 2: Assume true for $n = k$

$$\left(x^{3^k} - 1\right) = (x - 1)\left(x^2 + x + 1\right)\left(x^6 + x^3 + 1\right)\ldots\ldots\left(x^{2\left(3^{k-1}\right)} + x^{3^{k-1}} + 1\right)$$

Step 3: Prove true for $n = k + 1$

LHS $\left(x^{3^{k+1}} - 1\right)$

RHS $\quad (x - 1)\left(x^2 + x + 1\right)\left(x^6 + x^3 + 1\right)\ldots\ldots\left(x^{2\left(3^{k-1}\right)} + x^{3^{k-1}} + 1\right)\left(x^{2\left(3^k\right)} + x^{3^k} + 1\right)$

$$= \left(x^{3^k} - 1\right)\left(x^{2\left(3^k\right)} + x^{3^k} + 1\right)$$

$$= x^{3\left(3^k\right)} + x^{2\left(3^k\right)} + x^{3^k} - x^{2\left(3^k\right)} - x^{3^k} - 1$$

$$= x^{3^{k+1}} - 1$$

= RHS as required

(b) (i) $\frac{AB}{AD} = \sqrt{2}$ (similar triangles ABC, ADE)

$\frac{AB}{BG} = \sqrt{2}$ (similar triangles ABC, GBF)

$\frac{BC}{HC} = \sqrt{2}$ (similar triangles ABC, IHC)

$\frac{BC}{BF} = \sqrt{2}$ (similar triangles ABC, GBF)

Therefore HC = BF $\implies$ BH = FC $\implies$ DY = ZE

(ii) Let BC = $\sqrt{2}$ and DE = 1.

$\frac{HC}{HI} = \frac{BC}{AB}$ from simliar triangles so $\frac{HC}{BC} = \frac{HI}{AB} = \frac{1}{\sqrt{2}} \implies HC = \frac{BC}{\sqrt{2}} \implies HC = 1$

Then $BH = FC = \sqrt{2} - 1$

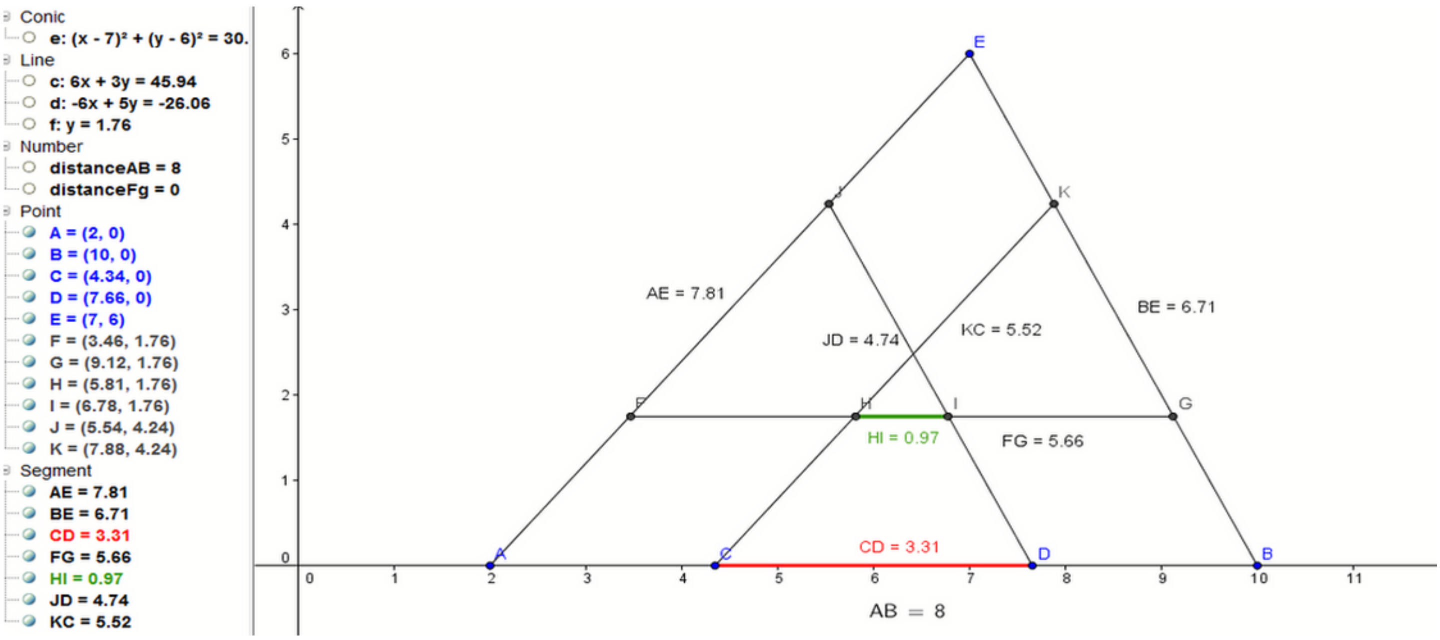
So $\quad HF = \sqrt{2} - 2\left(\sqrt{2} - 1\right) = 2 - \sqrt{2}$

Now $DY = ZE = BH = FC$ (BDYH and ECFZ are parallelograms) hence

$$YZ = 1 - 2\left(\sqrt{2} - 1\right) = 3 - 2\sqrt{2}$$

Therefore $\frac{YZ}{BC} = \frac{3-2\sqrt{2}}{\sqrt{2}} = \frac{3\sqrt{2}-4}{2}$

To check this I constructed an arbitrary triangle using *GeoGebra* with BC = 8 and the required parallel lines in the ratio $1:\sqrt{2}$. The diagram shows the co-ordinates of all points and the distances calculated by *GeoGebra*. Given BC = 8 then YZ should be $4(3\sqrt{2} - 4) \approx 0.97$. This is the distance shown in the diagram.



(c) We have $P(x) = x^3 + p\,x + q$

Note that the exam unfortunately used $p(x)$ for the polynomial and p for the coefficient of x.

Given the roots are α, β and γ we have

$$\alpha + \beta + \gamma = 0 \qquad \alpha\beta + \alpha\gamma + \beta\gamma = p \quad \text{and} \quad \alpha\beta\gamma = -q$$

(i) $(\beta - \gamma)^2 = \beta^2 + \gamma^2 - 2\,\beta\gamma$

Now $\quad (\alpha + \beta + \gamma)^2 = \alpha^2 + \beta^2 + \gamma^2 + 2\,(\alpha\beta + \alpha\gamma + \beta\gamma)$

So $\quad 0 = \alpha^2 + \beta^2 + \gamma^2 + 2\,p$

Therefore $\quad \beta^2 + \gamma^2 = -\alpha^2 - 2\,p$

From $\alpha\beta\gamma = -q$ we have $\beta\gamma = -\frac{q}{\alpha}$

Thus $\quad (\beta - \gamma)^2 = -\alpha^2 - 2\,p + 2\,\frac{q}{\alpha}$

Since $P(\alpha) = 0$ we have

$$\alpha^3 + p\,\alpha + q = 0 \implies p = -\alpha^2 - \frac{q}{\alpha}$$

Finally we have $\quad (\beta - \gamma)^2 = -\alpha^2 - 2\left(-\alpha^2 - \frac{q}{\alpha}\right) + 2\,\frac{q}{\alpha}$

$$= \alpha^2 + 4\,\frac{q}{\alpha} \qquad \text{as required}$$

(ii) $(\alpha - \beta)^2\,(\beta - \gamma)^2\,(\gamma - \alpha)^2$

Using (i) we can replace this expression by

$$\left(\gamma^2 + \frac{4\,q}{\gamma}\right)\left(\alpha^2 + \frac{4\,q}{\alpha}\right)\left(\beta^2 + \frac{4\,q}{\beta}\right)$$

Now $\quad \alpha^2 + \frac{4\,q}{\alpha} = \frac{\alpha^3 + 4\,q}{\alpha} = \frac{-p\,\alpha - q + 4\,q}{\alpha} = \frac{-p\,\alpha + 3\,q}{\alpha}$ using the fact that $\alpha^3 + p\,\alpha + q = 0$

Similarly for the other brackets to give:

$$\left(\frac{-p\,\gamma + 3\,q}{\gamma}\right)\left(\frac{-p\,\alpha + 3\,q}{\alpha}\right)\left(\frac{-p\,\beta + 3\,q}{\beta}\right)$$

Expanding and simplifying gives

$$-p^3 + 3\,p^2\,q\left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}\right) - 9\,p\,q^2\left(\frac{1}{\alpha\beta} + \frac{1}{\alpha\gamma} + \frac{1}{\beta\gamma}\right) + \frac{27\,q^3}{\alpha\beta\gamma}$$

$$= -p^3 + 3\,p^2\,q\left(\frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma}\right) - 9\,p\,q^2\left(\frac{\alpha + \beta + \gamma}{\alpha\beta\gamma}\right) + \frac{27\,q^3}{\alpha\beta\gamma}$$

Using the results given at the start of the question this reduces to

$$-p^3 - 3p^3 - 0 - 27q^2 \quad = \quad -\left(27q^2 + 4p^3\right) \quad \text{as required}$$

(iii) If $(27q^2 + 4p^3) < 0$ then $(\alpha - \beta)^2 (\beta - \gamma)^2 (\gamma - \alpha)^2 > 0$ and hence α, β and γ are all different.