Why is 0!=1?

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Students working with the factorial function for the first time may be confused why 0!=1. School students usually aren't told why the statement is correct and this note attempts to do so.

The Factorial Function

The factorial function is defined for positive integers n as

$$n! = n(n-1)(n-2) \ldots\ldots (3)(2)(1) \tag{1}$$

The function is not defined for non-integer positive numbers nor negative numbers.

The Gamma Function

The Gamma function extends the factorial function to non-integer positive and negative numbers. It is not defined for negative integers.

The definition of the Gamma function $\Gamma(x)$ for all complex numbers $x \neq 0, -1, -2, \ldots$ is

$$\Gamma(x) = \lim_{k \to \infty} \frac{k!\, k^{x-1}}{(x)_k} \tag{2}$$

where $(x)_k$ is the rising factorial:

$$(x)_k = x(x+1)(x+2) \ldots\ldots (x+k-1) \tag{3}$$

A useful alternative is, for $Re(x) > 0$,

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt \tag{4}$$

which gives

$$\Gamma(1) = \int_0^\infty e^{-t}\, dt = 1 \tag{5}$$

We then have, from either (2) or (4),

$$\Gamma(x+1) = x\,\Gamma(x) \tag{6}$$

We obtain this from (4) using integration by parts:

$$\Gamma(x+1) = \int_0^\infty t^x e^{-t}\, dt = \left[-t^x e^{-t}\right]_0^\infty + x \int_0^i t^{x-1} e^{-t}\, dt = x\,\Gamma(x) \tag{7}$$

Using (6) together with (5) gives $\Gamma(2) = 1\times\Gamma(1) = 1$; $\Gamma(3) = 2\times\Gamma(2) = 2$; $\Gamma(4) = 3\times\Gamma(3) = 3\times2 = 6$ and so on. Thus we see that

$$\Gamma(n) = (n-1)! \tag{8}$$

and hence, using $\Gamma(1) = 1$ from (5), we see that

$$\Gamma(1) = (1-1)! = 0! \tag{9}$$

ie we have the relationship $0! = 1$

Extending the Gamma function

From (6) we have

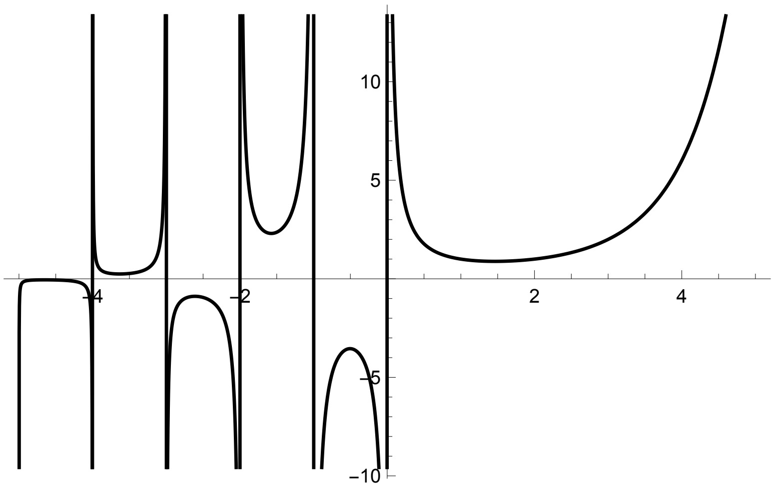
$$\Gamma(x) = \frac{\Gamma(x+1)}{x} \tag{10}$$

and so if we have Γ(x) for $0 < x < 1$ then we can calculate Γ(x) for $-1 < x < 0$.

For example

$$\Gamma\left(-\frac{1}{2}\right) = -2\,\Gamma\left(\frac{1}{2}\right) \tag{11}$$

The graph of Γ(x) is shown below:



Γ(x) for -5 < x < 5. Note asymptotes at *x* = -1, -2, -3,...

References

A. *Gamma,* Julian Havil, Princeton, 2003

B. *Special Functions,* George Andrews, Richard Askey and Ranjan Roy, Cambridge Universty Press, 1999