Phase portraits for systems of differential equations

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Consider the linear system of differential equations

$$y_1'(t) = a\,y_1(t) + b\,y_2(t)$$

$$y_2'(t) = c\,y_1(t) + d\,y_2(t)$$

or, in matrix form,

$$\boldsymbol{y}'(t) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \boldsymbol{y}(t)$$

Critical points occur when $\boldsymbol{y}'(t) = 0$ ie when $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \boldsymbol{y}(t) = 0$. The type and stability of critical point depends upon the values of the pair of eigenvalues $(\lambda_1,\ \lambda_2)$ of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The possible combinations are given in the table.

Eigenvalues	**Type of critical point**	**Stability**
Both positive, different	Improper node	Unstable
Both negative, different	Improper node	Asymptotically stable
One positive, one negative	Saddle point	Unstable
Equal and positive (independent eigenvectors)	Proper node	Unstable
Equal and positive (dependent eigenvectors)	Improper node	Unstable
Equal and negative (independent eigenvectors)	Proper node	Asymptotically stable
Equal and negative (dependent eigenvectors)	Improper node	Asymptotically stable
Complex conjugates with positive real part	Spiral point	Unstable
Complex conjugates with negative real part	Spiral point	Asymptotically stable
Purely complex conjugates	Centre	Stable

The following pages show examples of each type.

The phase portraits illustrate how $\frac{dy_2(t)}{dy_1(t)}$ changes according to the values of $y_1(t)$ (horizontal axis) and $y_2(t)$ (vertical axis).

(1) Unequal positive eigenvalues

The matrix $\begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}$ has eigenvalues λ = (2,4) and the critical point (0,0) is an unstable improper node, as shown in the phase portrait. Note that all streamlines move out from the origin.

The *Mathematica* code is:

```
StreamPlot[{5 x - y, 3 x + y}, {x, -3, 3}, {y, -3, 3}, PlotTheme → "Scientific"]
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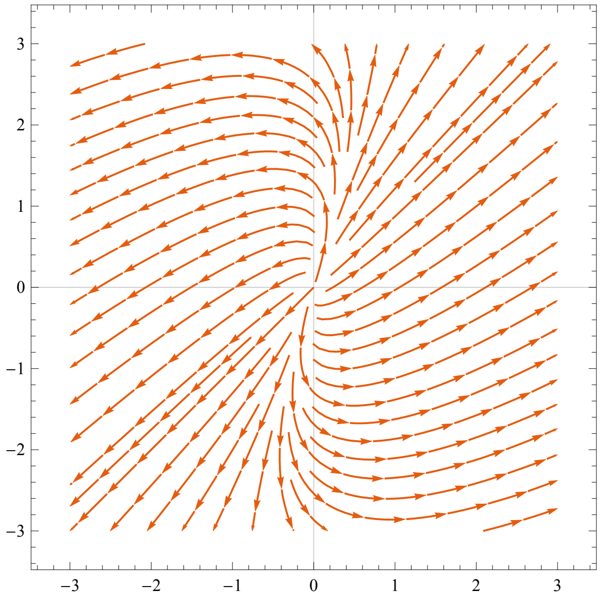
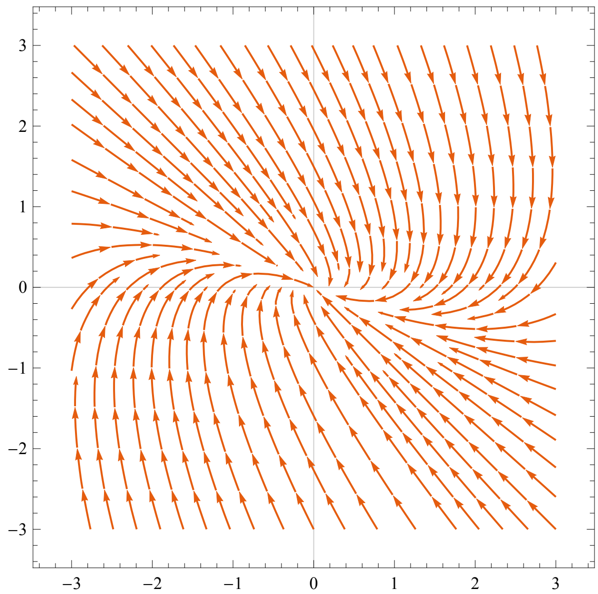


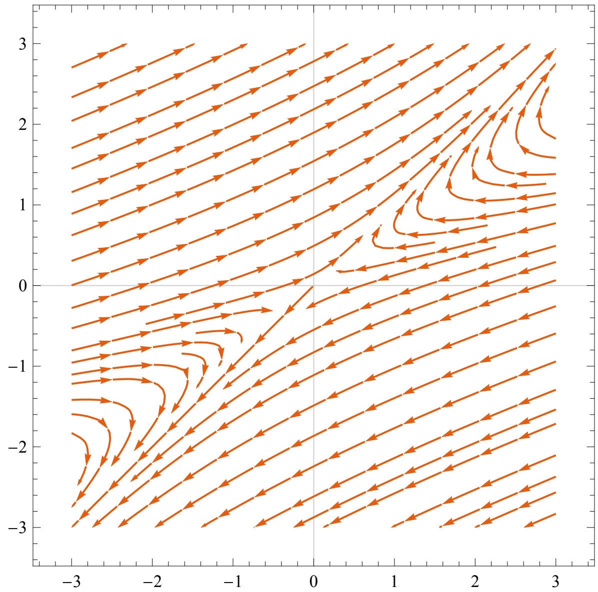
(2) Unequal negative eigenvalues

The matrix $\begin{pmatrix} -1 & 2 \\ -1 & -4 \end{pmatrix}$ has eigenvalues λ = (-3,-2) and the critical point (0,0) is an asymptotically stable improper node as shown in the phase portrait. Note that all streamlines move towards the origin.



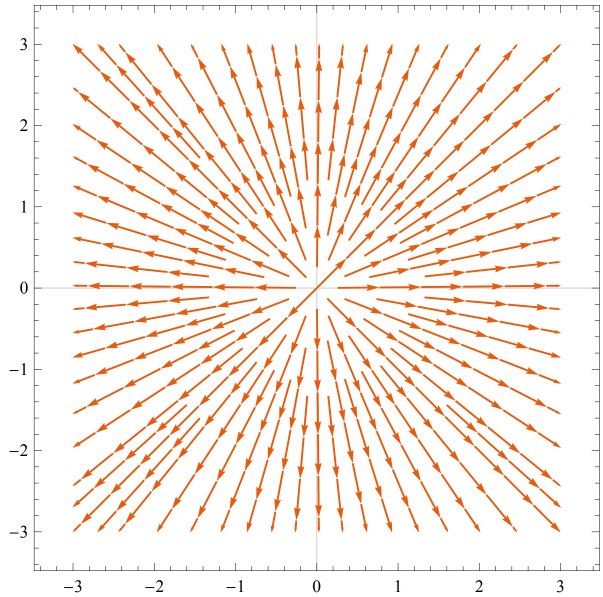
(3) One positive and one negative eigenvalue

The matrix $\begin{pmatrix} -3 & 4 \\ -1 & 2 \end{pmatrix}$ has eigenvalues λ = (-2,1) and the critical point (0,0) is an unstable saddle point, as shown in the phase portrait. Note that all streamlines move towards and then away from the origin.



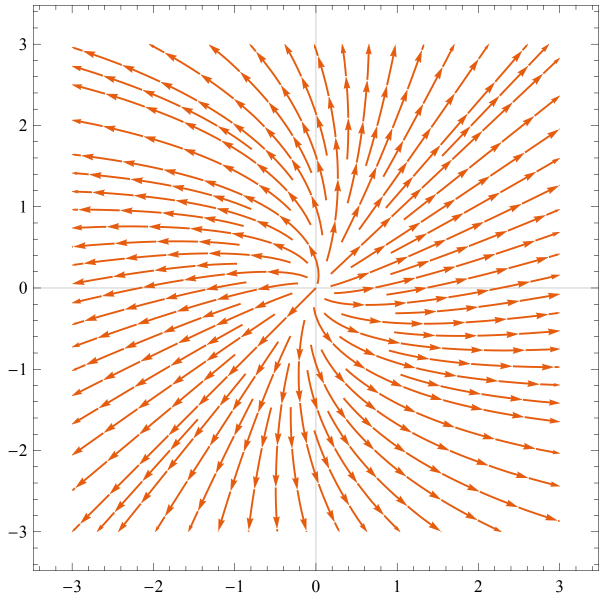
(4) Equal positive eigenvalues with independent eigenvectors

The matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ has eigenvalues λ = (1,1) with eigenvectors $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. The critical point (0,0) is an unstable proper node, as shown in the phase portrait. Note that all streamlines move away from the origin.



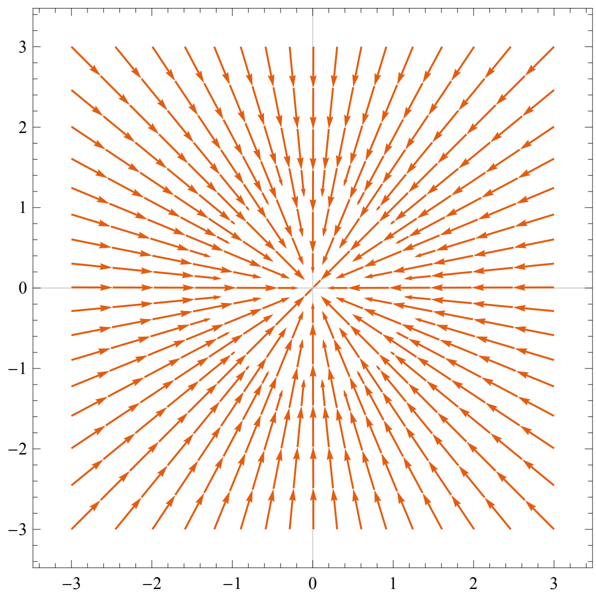
(5) Equal positive eigenvalues with dependent eigenvectors

The matrix $\begin{pmatrix} 5 & -1 \\ 1 & 3 \end{pmatrix}$ has repeated eigenvalue $\lambda = 4$ with dependent eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$. The critical point (0,0) is an unstable improper node, as shown in the phase portrait. Note that all streamlines move away from the origin.



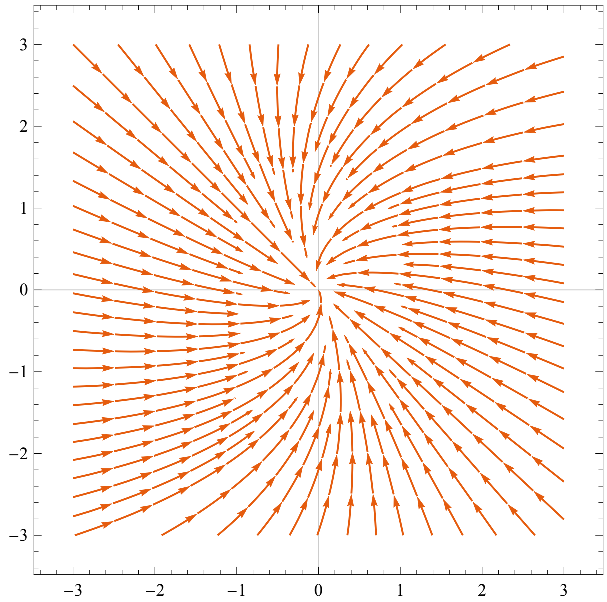
(6) Equal negative eigenvalues with independent eigenvectors

The matrix $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ has eigenvalues λ = (-1,-1) with eigenvectors $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. The critical point (0,0) is an asymtotically stable proper node, as shown in the phase portrait. Note that all streamlines move toward the origin.



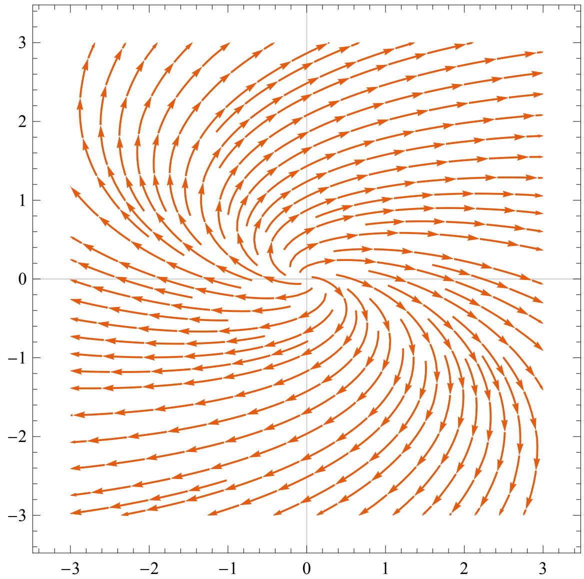
(7) Equal negative eigenvalues with dependent eigenvectors

The matrix $\begin{pmatrix} -5 & -1 \\ 1 & -3 \end{pmatrix}$ has repeated eigenvalue λ = -4 with dependent eigenvectors $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$. The critical point (0,0) is an asymtotically stable improper node, as shown in the phase portrait. Note that all streamlines move toward the origin.



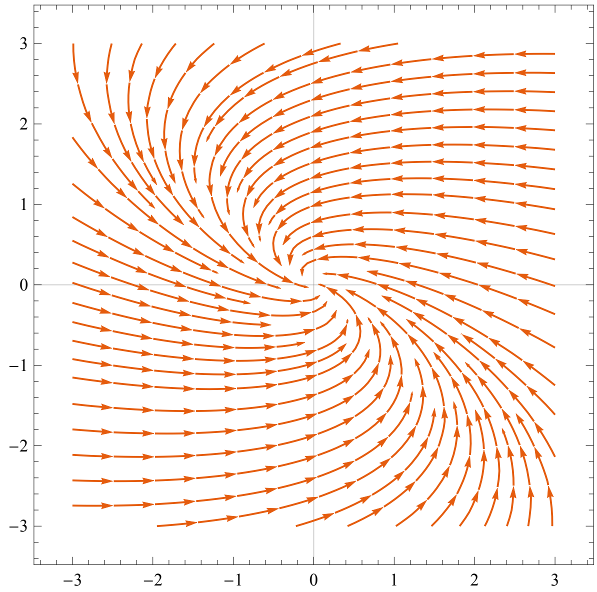
(8) Complex conjuate eigenvalues with positive real part

The matrix $\begin{pmatrix} 3 & 4 \\ -1 & 2 \end{pmatrix}$ has eigenvalues $\lambda = \left(\frac{5+i\sqrt{15}}{2}, \ \frac{5-i\sqrt{15}}{2} \right)$ and the critical point (0,0) is an unstable spiral point, as shown in the phase portrait. Note that all streamlines move away from the origin.



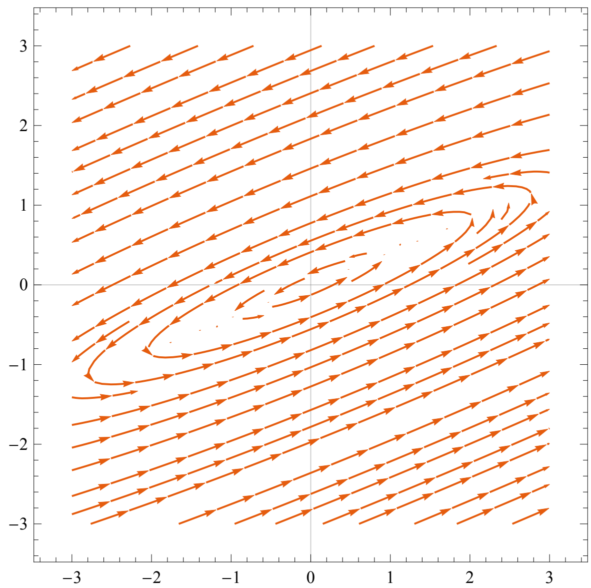
(9) Complex conjuate eigenvalues with negative real part

The matrix $\begin{pmatrix} -3 & -3 \\ 1 & -1 \end{pmatrix}$ has eigenvalues $\lambda = (-2 + i\sqrt{2},\ -2 - i\sqrt{2})$ and the critical point (0,0) is an asymptotically stable spiral point, as shown in the phase portrait. Note that all streamlines move toward the origin.



(10) Purely complex conjuate eigenvalues

The matrix $\begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}$ has eigenvalues $\lambda = (i,\ -i)$ and the critical point (0,0) is a stable centre point, as shown in the phase portrait. Note that all streamlines are centred around the origin with anti-clockwise rotation.



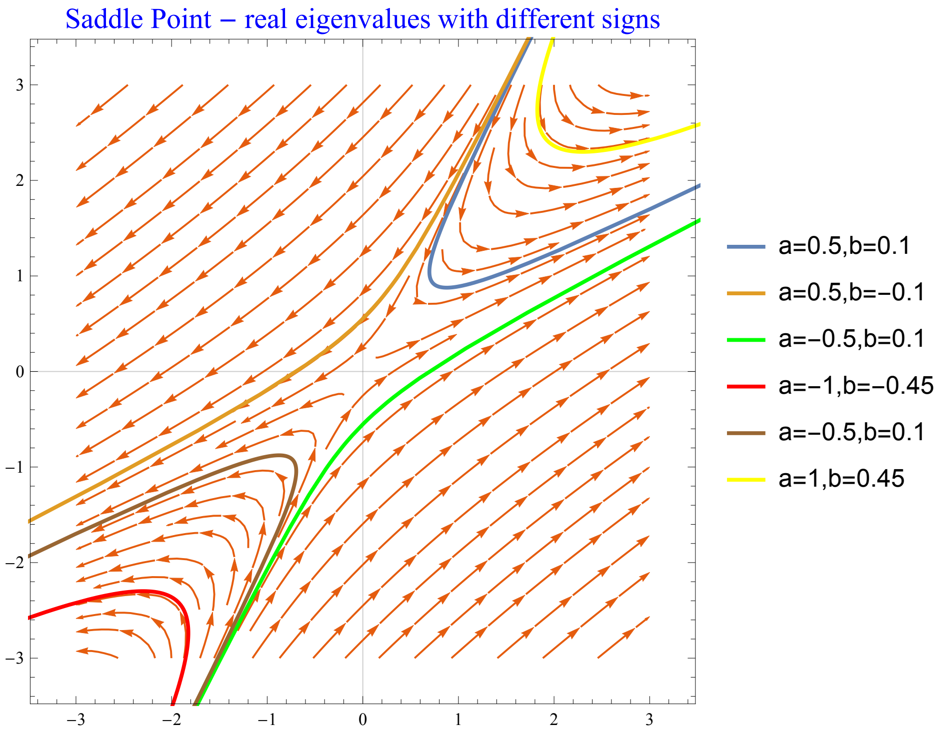
A closer look at one linear system

The matrix $\begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$ has eigenvalues λ = -1 and 2 with independent eigenvectors $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$. The critical point (0,0) is an saddlepoint.

The solutions to the system of differential equations are

$$y^{(1)}(t) = a \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t} \qquad \text{and} \qquad y^{(2)}(t) = b \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t}$$

where *a* and *b* are arbitrary constants, depending upon the initial conditions. Each combination of *a* and *b* produces a trajectory. The following plot shows the streamlines with a sample of trajectories. Note that $y^{(1)}(t)$ is on the horizontal axis and $y^{(2)}(t)$ on the vertical axis.



An example of a non-linear system: the predator / prey model.

A simple non-linear system models the situation where predators have prey as a source of food while the prey have a different source of food.

If y_1 is the prey population and y_2 the predator population then the system is

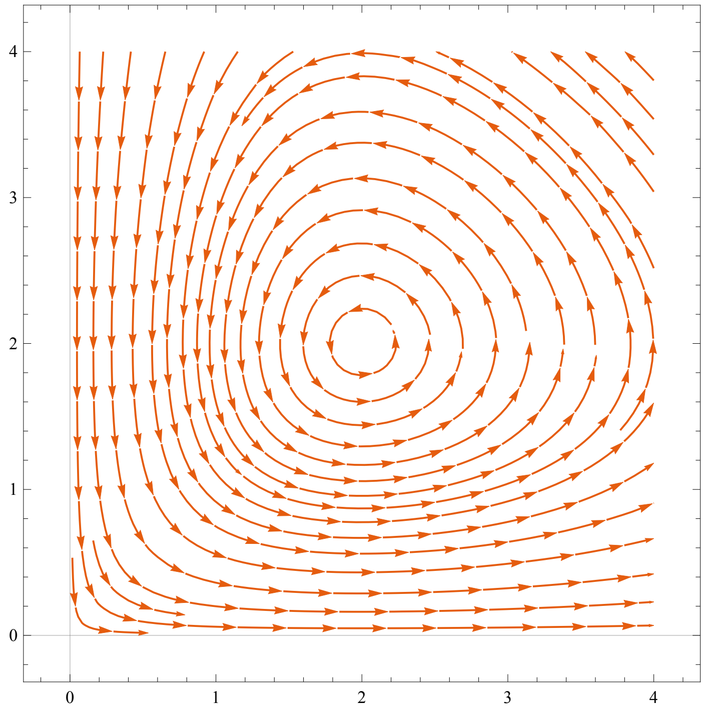
$$y_1'(t) = a\, y_1(t) - b\, y_1(t)\, y_2(t)$$

$$y_2'(t) = -c\, y_2(t) + d\, y_1(t)\, y_2(t)$$

where *a, b, c* and *d* are all positve. These are Lotka-Volterra equations where both populations are simultaneously subject to increasing and decreasing forces.

Here is the phase-portrait when a = 1, b = 0.5, c = 1 and d = 0.5:

```
StreamPlot[{x (1 - 0.5 y), y (-1 + 0.5 x)},
 {x, 0, 4}, {y, 0, 4}, PlotTheme → "Scientific"]
```



Equilibrium exists at (2,2). In the top right quadrant the increasing predator population causes the prey population to decrease. This increasing shortage of prey causes the predator population to decrease (top left quadrant). The result of this is an increase in prey population (bottom left), thus causing the small predator population to grow (bottom right).