Deriving an expression for the general term of a sequence from the sum and vice versa

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An exam question asked to find an expression for the general term in a sequence where the sum of the first n terms is given by $S_n = \frac{n}{2(n+2)}$.

This is an example of a more general formula $S_n = \frac{n}{(k+1)(n+k+1)}$

Solution

$$T_n = S_n - S_{n-1}$$

$$= \frac{n}{(k+1)(n+k+1)} - \frac{n-1}{(k+1)(n+k)}$$

$$= \frac{1}{k+1}\left(\frac{n(n+k) - (n-1)(n+k+1)}{(n+k)(n+k+1)}\right)$$

$$= \frac{1}{k+1}\left(\frac{k+1}{(n+k)(n+k+1)}\right)$$

$$= \frac{1}{(n+k)(n+k+1)}$$

$$= \frac{1}{n+k} - \frac{1}{n+k+1}$$

Note that as $\frac{1}{(n+k)(n+k+1)} = \frac{1}{n^2+(2k+1)n+(k^2+k)}$ we can now look at deriving the sum from the general term.

Sum from general term

Given $T_n = \frac{1}{n^2+(2k+1)n+(k^2+k)}$ show that $\sum_{r=1}^{n} \frac{1}{r^2+(2k+1)r+(k^2+k)} = \frac{n}{(k+1)(n+k+1)}$

$$T_n = \frac{1}{n^2 + (2k+1)n + (k^2+k)}$$

$$= \frac{1}{n^2 + (2k+1)n + k(k+1)}$$

$$= \frac{1}{(n+k)(n+k+1)}$$

$$= \frac{1}{n+k} - \frac{1}{n+k+1}$$

Now $\quad S_n = \sum_{r=1}^{n} T_r$

$$= \left(\frac{1}{1+k} - \frac{1}{2+k}\right) + \left(\frac{1}{2+k} - \frac{1}{3+k}\right) + \ldots\ldots + \left(\frac{1}{n-1+k} - \frac{1}{n+k}\right) + \left(\frac{1}{n+k} - \frac{1}{n+k+1}\right)$$

$$= \frac{1}{1+k} - \frac{1}{n+k+1}$$

$$= \frac{n}{(k+1)(n+k+1)}$$