Fourier Series example

Dr Richard Kenderdine
Kenderdine Maths Tutoring

The Fourier Series for a function $f(x)$ defined on $-L \le x \le L$ is given by

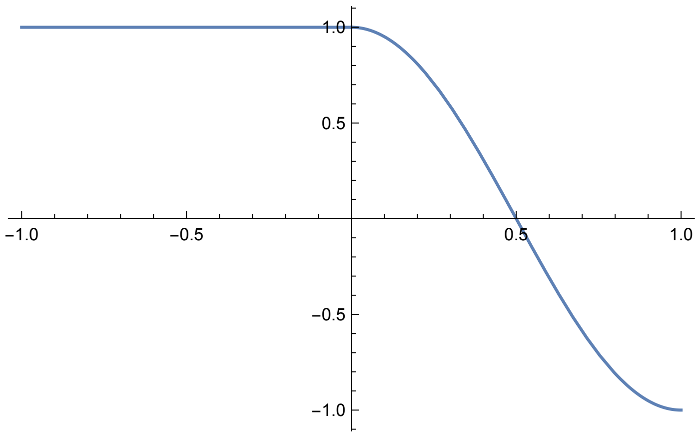
$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left[\frac{n\pi x}{L}\right] + b_n \sin\left[\frac{n\pi x}{L}\right] \right)$$

where

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\, dx \qquad a_n = \frac{1}{L}\int_{-L}^{L} f(x) \cos\left[\frac{n\pi x}{L}\right] dx \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin\left[\frac{n\pi x}{L}\right] dx$$

Now consider the following function:

$$f(x) = \begin{cases} 1 & -1 \le x \le 0 \\ \cos[\pi x] & 0 < x \le 1 \end{cases}$$



Calculation of a_0 and a_n $(n > 1)$:

$$a_0 = \frac{1}{2}\left[\int_{-1}^{0} 1\, dx + \int_{0}^{1} \cos[\pi x]\, dx\right] = \frac{1}{2}$$

$$a_n = \int_{-1}^{0} \cos[n\pi x]\, dx + \int_{0}^{1} \cos[\pi x] \cos[n\pi x]\, dx$$

$$= \left[\frac{\sin[n\pi x]}{n\pi}\right]_{-1}^{0} + \int_{0}^{1} \cos[\pi x] \cos[n\pi x]\, dx$$

$$= 0 + 0 = 0 \qquad (\text{for } n > 1)$$

When $n = 1$ we have

$$a_1 = \int_{0}^{1} \cos^2[\pi x]\, dx = \frac{1}{2}$$

Note that a_0 calculates the average function value over the interval. We use the orthogonality property that $\int_0^1 \text{Cos}[\pi x]\,\text{Cos}[n\,\pi\,x]\,dx = 0$ for $n>1$.

Now calculate the b_n values (using Sin(A)Cos(B) = $\frac{1}{2}[\text{Sin}(A+B)+\text{Sin}(A-B)]$):

$$b_n = \int_{-1}^{0} \text{Sin}[n\,\pi\,x]\,dx + \int_0^1 \text{Cos}[\pi\,x]\,\text{Sin}[n\,\pi\,x]\,dx$$

$$= \left[\frac{-\text{Cos}[n\,\pi\,x]}{n\pi}\right]_{-1}^{0} + \frac{1}{2}\int_0^1 \left[[\text{Sin}[(n+1)\,\pi\,x] + \text{Sin}[(n-1)\,\pi\,x]]\,dx\right.$$

$$= \frac{1}{n\pi}[-1+(-1)^n] + \frac{1}{2}\left[\frac{-\text{Cos}[(n+1)\,\pi\,x]}{(n+1)\,\pi} - \frac{\text{Cos}[(n-1)\,\pi\,x}{(n-1)\,\pi}\right]_0^1$$

$$= \frac{1}{n\pi}[-1+(-1)^n] + \frac{1}{2\,\pi}\left[\frac{-\text{Cos}[(n+1)\,\pi]}{(n+1)} - \frac{\text{Cos}[(n-1)\,\pi}{(n-1)} + \frac{1}{(n+1)} + \frac{1}{(n-1)}\right]$$

The value of b_n depends upon whether n is odd or even.

$$\frac{1}{n\pi}[-1+(-1)^n] = \begin{cases} \frac{-2}{n\pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

$$\frac{1}{2\,\pi}\left[\frac{-\text{Cos}[(n+1)\,\pi]}{(n+1)} - \frac{\text{Cos}[(n-1)\,\pi}{(n-1)} + \frac{1}{(n+1)} + \frac{1}{(n-1)}\right]$$
$$= \begin{cases} 0 & \text{if } n \text{ is odd} \\ \frac{1}{\pi}\left[\frac{1}{n+1} + \frac{1}{n-1}\right] & \text{if } n \text{ is even} \end{cases}$$

If n is even then both n - 1 and n + 1 are odd. To ensure both are odd, use $2n$ - 1 and $2n$ + 1 and then

$$\frac{1}{\pi}\left[\frac{1}{2\,n+1} + \frac{1}{2\,n-1}\right] = \frac{1}{\pi}\left[\frac{4\,n}{4\,n^2-1}\right]$$

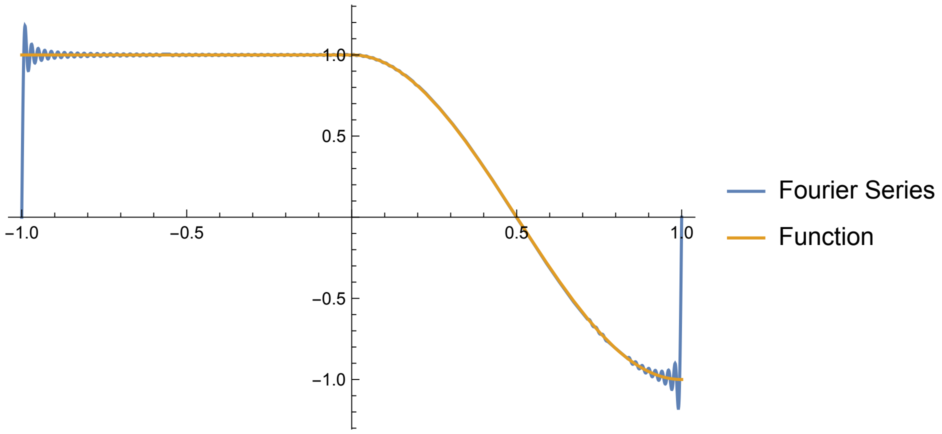
Also, we had $\frac{-2}{n\,\pi}$ when n is odd. To ensure the index is odd, use $2n$ - 1.

Now collecting it all together, we have

$$f\,(x) = \frac{1}{2} + \frac{1}{2}\,\text{Cos}[\pi\,x] + \sum_{n=1}^{\infty}\left[\frac{4\,n}{\left(4\,n^2-1\right)\,\pi}\,\text{Sin}[2\,n\,\pi\,x] - \frac{2}{(2\,n-1)\,\pi}\,\text{Sin}[(2\,n-1)\,\pi\,x]\right]$$

Here is a plot of the Fourier Series when n = 50:

Out[7]=



Increasing the number of terms will damp down the fluctuations at the endpoints; here is the plot when n = 200:

Out[6]=