An interesting complex locus problem

Dr Richard Kenderdine
Kenderdine Maths Tutoring

The problem is to find the maximum point z lying on the locus of points that satisfy

$$\left| z + \frac{1}{z} \right| = a \tag{1}$$

for some constant a, where z is a complex number.

Let $z = r\,e^{i\theta} = r\,(\cos\theta + i\sin\theta)$ then $\frac{1}{z} = \frac{1}{r}\,e^{-i\theta} = \frac{1}{r}\,(\cos\theta - i\sin\theta)$

and
$$z + \tfrac{1}{z} = \left(r + \tfrac{1}{r}\right)\cos\theta + i\left(r - \tfrac{1}{r}\right)\sin\theta$$

Hence
$$\left| z + \tfrac{1}{z} \right|^2 = \left(r + \tfrac{1}{r}\right)^2 \cos^2\theta + \left(r - \tfrac{1}{r}\right)^2 \sin^2\theta$$

$$= \left(r^2 + \tfrac{1}{r^2}\right)\left(\cos^2\theta + \sin^2\theta\right) + 2\left(\cos^2\theta - \sin^2\theta\right)$$

$$= \left(r^2 + \tfrac{1}{r^2}\right) + 2\cos 2\theta$$

Now the locus equation becomes

$$\left(r^2 + \frac{1}{r^2}\right) + 2\cos 2\theta = a^2 \tag{2}$$

Multiplying by r^2 yields a quadratic in r^2:

$$r^4 + \left(2\cos 2\theta - a^2\right) r^2 + 1 = 0 \tag{3}$$

Using the quadratic formula provides solutions for r^2 :

$$r^2 = \frac{-\left(2\cos 2\theta - a^2\right) \pm \sqrt{\left(2\cos 2\theta - a^2\right)^2 - 4}}{2} \tag{4}$$

We have r defined in terms of θ, a polar equation. Rembering that r is the modulus of z, maximising r will maximise z. Now $-\left(2\cos 2\theta - a^2\right) = a^2 - 2\cos 2\theta$ which has maximum value $a^2 + 2$ when $\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$ and if we take the positive square root, again to maximise r, we have

$$r^2 = \frac{\left(a^2 + 2\right) + \sqrt{\left(a^2 + 2\right)^2 - 4}}{2} \tag{5}$$

Upon simplifying we have

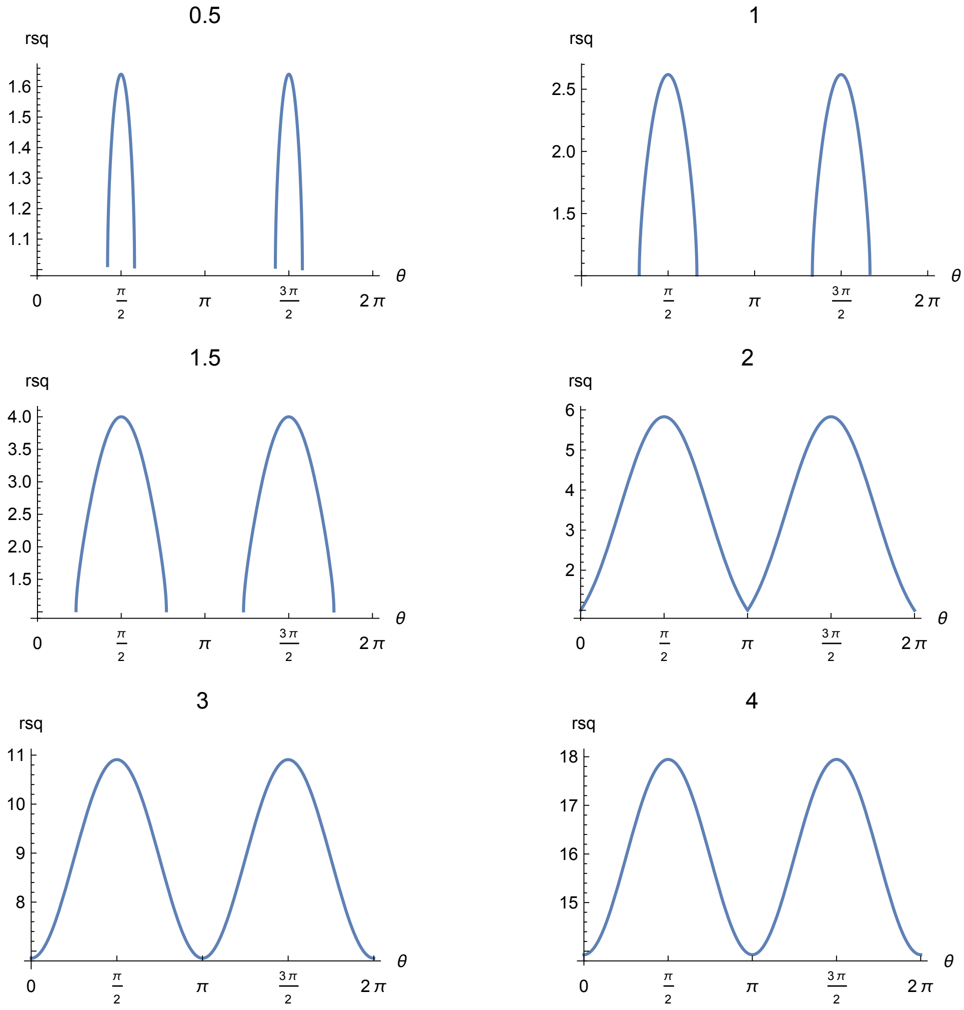
$$r^2 = \frac{\left(a^2 + 2\right) + a\sqrt{a^2 + 4}}{2} \tag{6}$$

and hence

$$r = \sqrt{\frac{(a^2+2) + a\sqrt{a^2+4}}{2}} \tag{7}$$

So the maximum value of z occurs at this value of r and when $\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$.

The following plots show r^2 for a = 0.5, 1, 1.5, 2, 3 and 4. As can be seen, the maximum values of r^2 occur when $\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$ (90° or 270°)



Plotting the locus

What does a plot of the locus of points satisfying $\left|z + \frac{1}{z}\right| = a$ look like? To find out we now use Cartesian co-ordinates.

Let $z = x + iy$. Then

$$\left|z + \frac{1}{z}\right| = \left|x + iy + \frac{1}{x+iy}\right| = \left|x + iy + \frac{1}{x+iy}\,\frac{x-iy}{x-iy}\right| = \left|x + iy + \frac{x-iy}{x^2+y^2}\right| \tag{8}$$

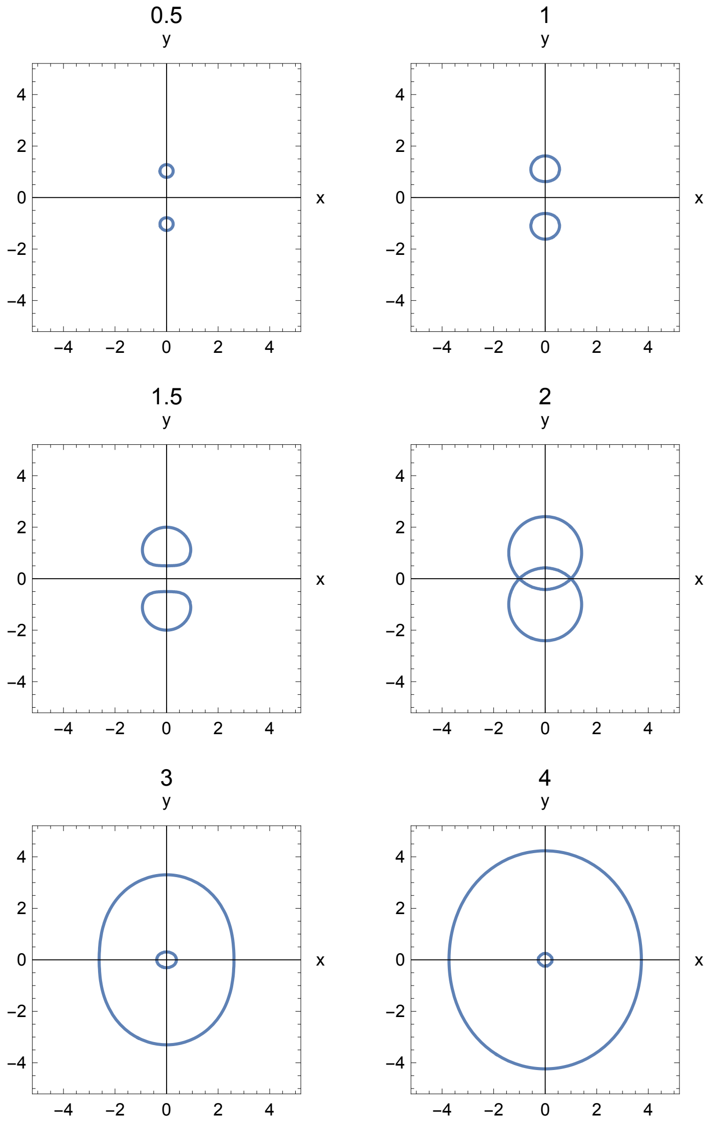
and

$$\left|z + \frac{1}{z}\right|^2 = \left(x + \frac{x}{x^2+y^2}\right)^2 + \left(y - \frac{y}{x^2+y^2}\right)^2 = x^2\left(1 + \frac{1}{x^2+y^2}\right)^2 + y^2\left(1 - \frac{1}{x^2+y^2}\right)^2 \tag{9}$$

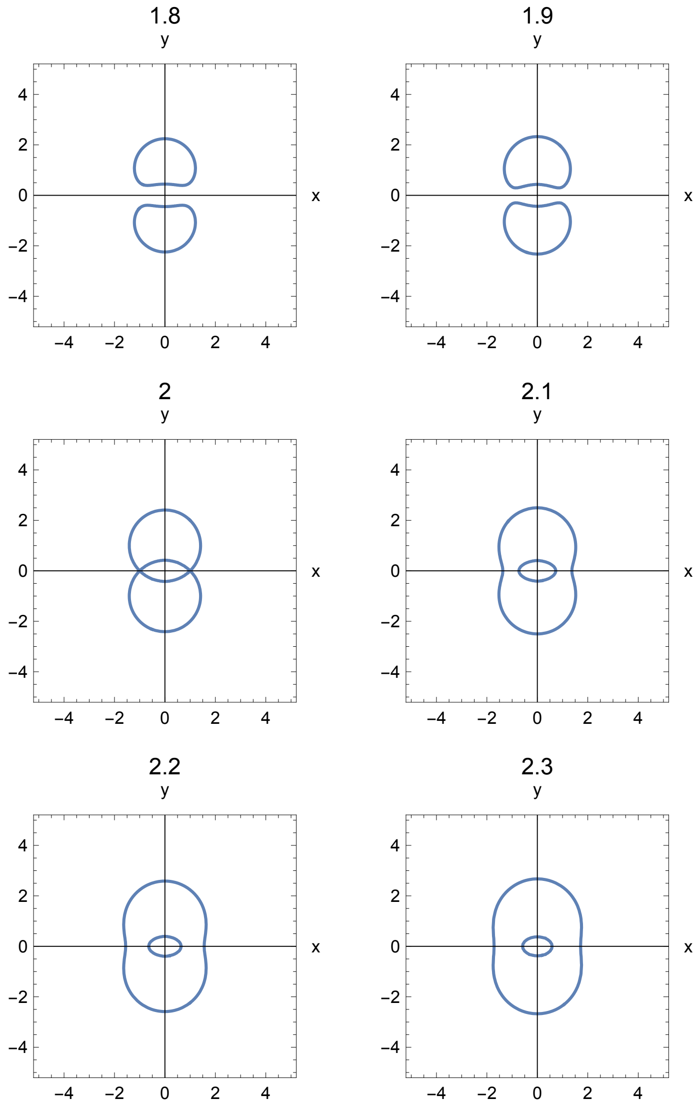
We then plot, for some values of *a,* the function defined by:

$$x^2\left(1+\frac{1}{x^2+y^2}\right)^2+y^2\left(1-\frac{1}{x^2+y^2}\right)^2=a^2 \tag{10}$$

The following plots show the function for a = 0.5, 1, 1.5, 2, 3 and 4:



These plots are interesting. If we only looked at the plots for a = 1, 2 and 4 then it would appear that the locus consists of two circles, initially the same size, that come together then one circle increases while the other decreases. However the plot for a =1.5 indicates something more is happening. The plots below are for a = 1.8 - 2.3:



We see that the flattening out of the circle that occurred for $a = 1.5$ changes to a concave section as a approaches 2. When $a = 2$ the two parts of the curve touch and as a increases beyond 2 the concave sections join and separate from the surrounding part of the curve. The two shapes now resemble an ellipse and a nephroid. For larger values of a the curve approaches small and large concentric circles.

Why does this happen? If we use Eqn (10) to find the x-intercepts, by setting $y = 0$, we obtain

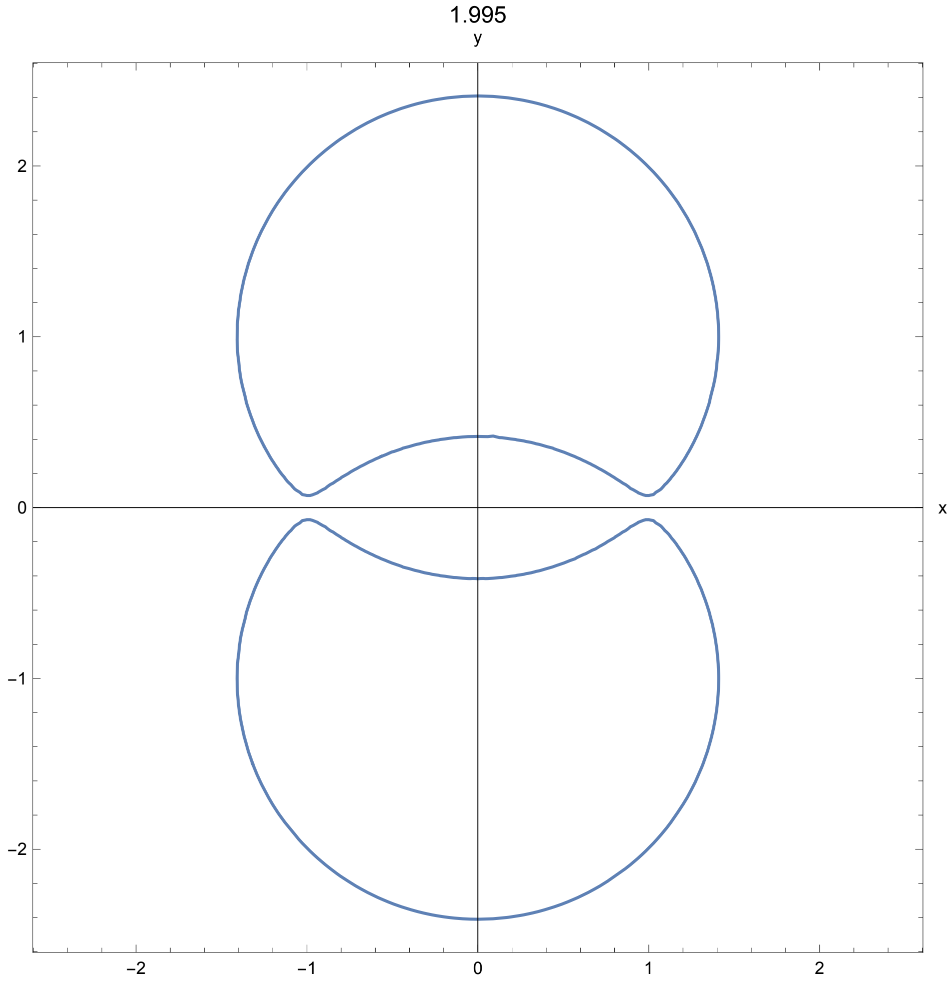
$$x = \pm \sqrt{\frac{(a^2 - 2) \pm a \sqrt{a^2 - 4}}{2}} \tag{11}$$

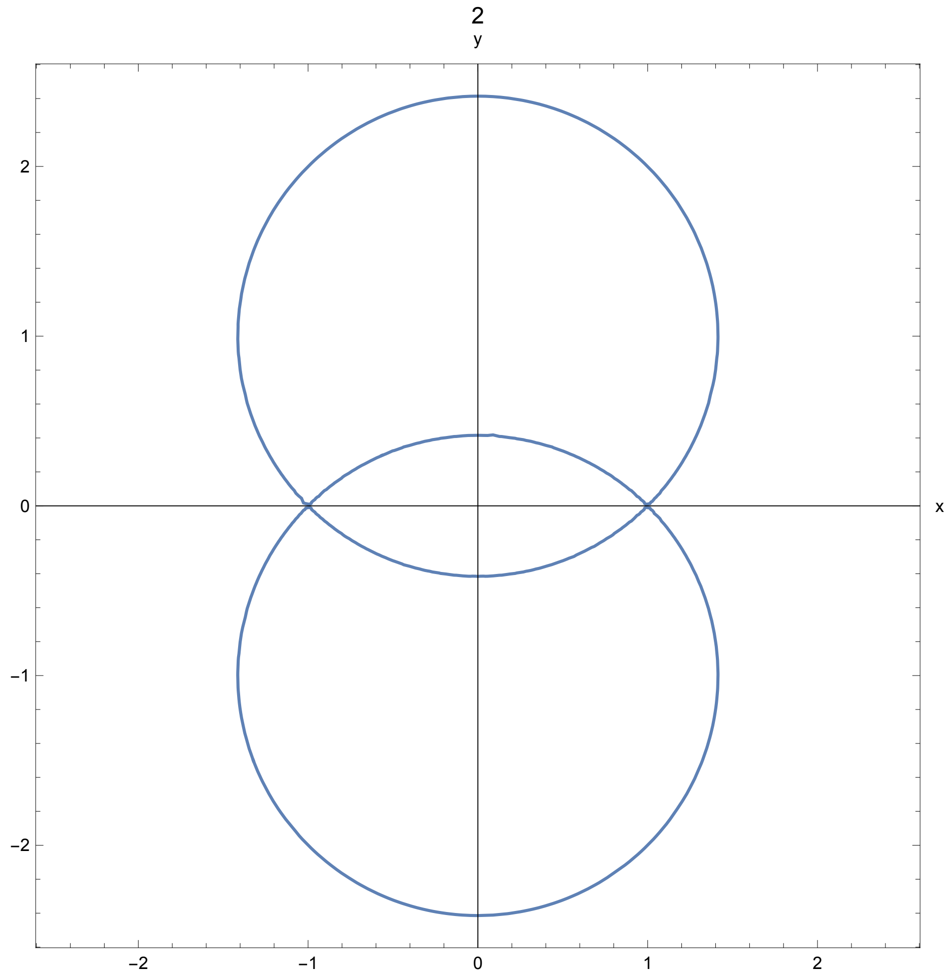
which is equivalent to Eqn (4) with $\theta = 0$. From (11) we see that there are no x-intercepts when $a < 2$, two intercepts when $a = 2$ (at $x = \pm 1$) and four intercepts when $a > 2$.

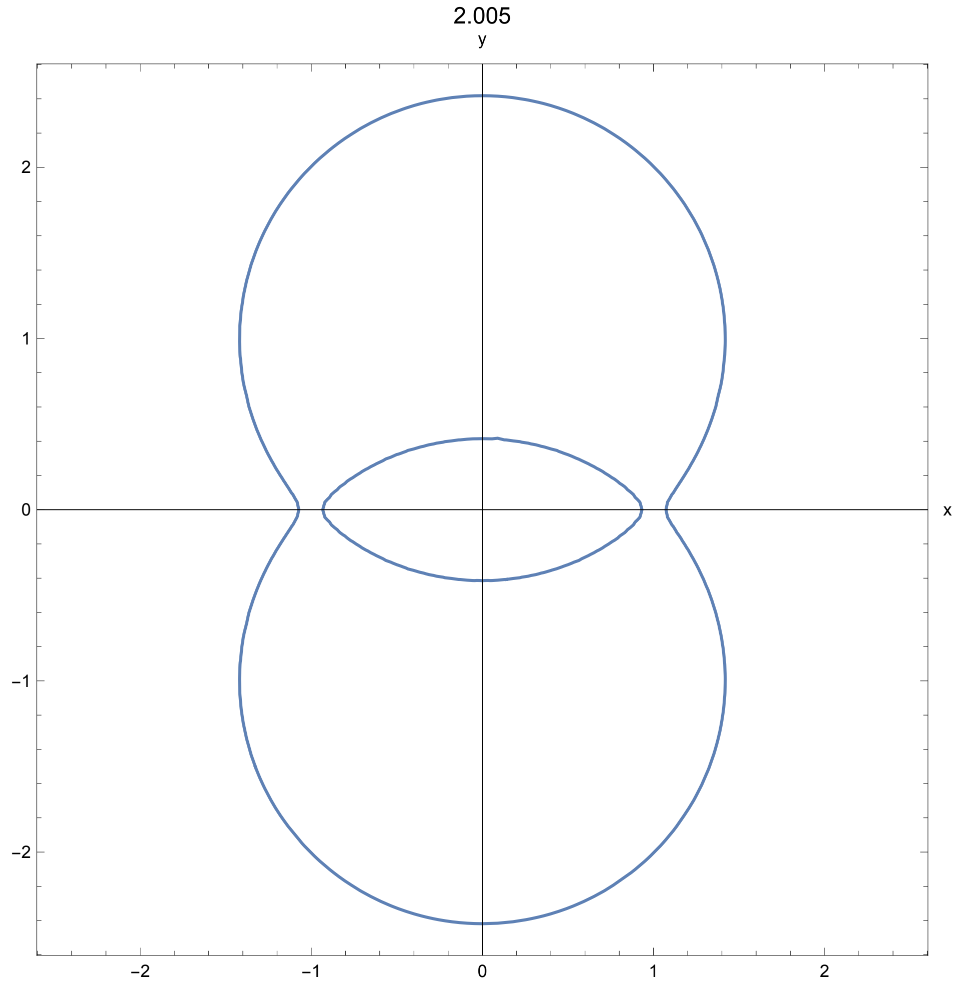
The y-intercepts are similarly obtained from (10) by setting $x = 0$. The resulting equation is equivalent to Eqn (7)

$$y = \pm \sqrt{\frac{(a^2 + 2) + a \sqrt{a^2 + 4}}{2}} \tag{12}$$

Enlarged plots for *a* = 1.995, 2 and 2.005 are shown:




2
y
x
2
1
0
−1
−2
−2
−1
0
1
2


2.005
y
x
2
1
0
−1
−2
−2
−1
0
1
2