2016 Mathematics Extension 1: Q14(c)

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The equation of the tangent at $T(2at, at^2)$ is $y = tx - at^2$ and to find the co-ordinates of the point D we let $y = -a$. Then $x = a(t - \frac{1}{t})$.

The equation of the normal at T is

$$y = -\frac{1}{t}(x - 2at) + at^2 \qquad (1)$$

Substituting $x = a(t - \frac{1}{t})$ gives $y = a(t + \frac{1}{t})^2 - a$

To find the Cartesian equation linking x and y we need to eliminate the parameter t. The first step is to expand x^2 and compare with y:

$$x^2 = a^2(t - \tfrac{1}{t})^2 = a^2\left(t^2 - 2 + \tfrac{1}{t^2}\right)$$

so $\frac{x^2}{a^2} = t^2 - 2 + \frac{1}{t^2}$

and $\frac{y}{a} = t^2 + 2 + \frac{1}{t^2} - 1 = t^2 + 1 + \frac{1}{t^2}$

thus $\frac{x^2}{a^2} + 3 = \frac{y}{a}$

and finally $x^2 = a(y - 3a)$ is the equation of the parabola P_2

The point of intersection of the normal at T with parabola P_2 is denoted R. From above, the co-ordinates of R are $(a(t - \frac{1}{t}), a(t + \frac{1}{t})^2 - a)$. We are told that the minimum distance between T and R occurs when the normals at T and R coincide. For this to occur the gradients of the normals at the two points must be the same which also means that the gradients of the tangents are equal.

We know the gradient of the tangent at T is t; the gradient of the tangent at R is found thus:

Since $x^2 = a(y - 3a)$ is the equation of the parabola P_2

then $y = \frac{x^2}{a} + 3a$

and so $y' = \frac{2x}{a}$

and at R $y' = \frac{2}{a}[a(t - \frac{1}{t})] = 2(t - \frac{1}{t})$

Now equating gradients $t = 2(t - \frac{1}{t}) \implies t - \frac{2}{t} = 0 \implies t^2 - 2 = 0 \implies t = \pm\sqrt{2}$

Thus the mimimum distance between T and R occurs when $t = \pm\sqrt{2}$

Solution by differentiation

We can also obtain this solution by analysing the distance function:

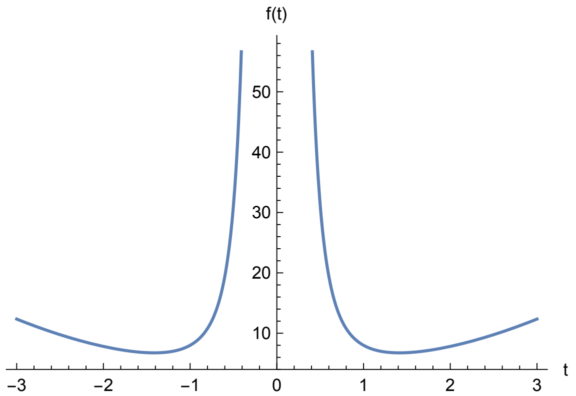
First simplify the distances $(x_T - x_R)$ and $(y_T - y_R)$ to be $a\,t + \frac{a}{t}$ and $-\left(a + \frac{a}{t^2}\right)$ respectively. Then

$$d(TR) = \sqrt{\left(a\,t + \frac{a}{t}\right)^2 + \left(a + \frac{a}{t^2}\right)^2} = a\sqrt{\left(t + \frac{1}{t}\right)^2 + \left(1 + \frac{1}{t^2}\right)^2}$$

Now square this expression and ignore a, since it is constant, expand and collect like terms to give:

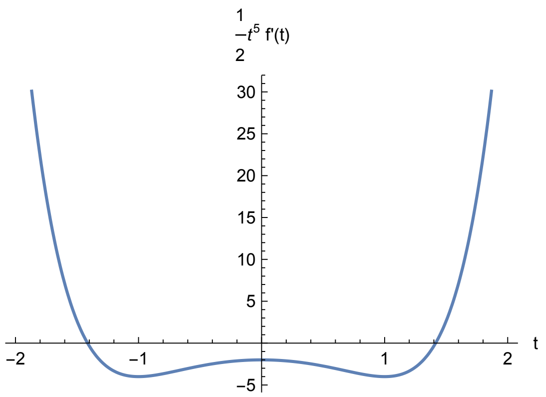
$$f(t) = t^2 + 3 + \frac{3}{t^2} + \frac{1}{t^4}$$

The plot of this function shows two mimimum points:

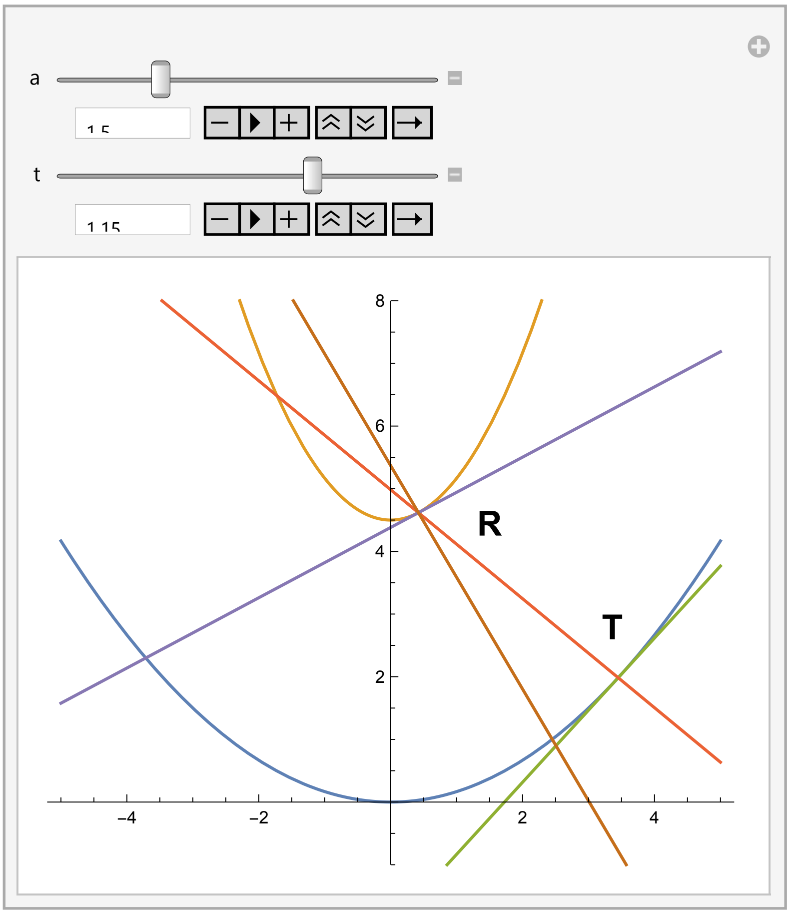


We can locate these points by setting $f'(t) = 0$:

$f'(t) = 2t - \frac{6}{t^3} - \frac{4}{t^5} = 0 \implies t^6 - 3\,t^2 - 2 = 0$ which has $\pm\sqrt{2}$ as the only real solutions, as shown in the plot of $\frac{1}{2}\,t^5 f'(t)$:



Here is a plot of the two parabolas, the points *T* and *R* and the tangents and normals at the respective points. The values of *a* and *t* are 1.5 and 1.15 respectively. Note that the normals do not coincide and hence the distance *TR* is not minimised.



Now the value of *t* is increased to $\sqrt{2}$ and we see that the normals coincide, hence the distance *TR* is minimised.