Connection between two standard integrals

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The standard integrals $\int \frac{1}{\sqrt{1-x^2}}\, dx = \sin^{-1} x$ and $\int \frac{1}{\sqrt{x^2+1}}\, dx = \text{Log}[x + \sqrt{x^2+1}\,]$ appear to be unrelated but this can be shown to not be the case.

The complex exponential and hyperbolic functions

We have the identity

$$e^{i\theta} = \cos\theta + i \sin\theta \tag{1}$$

and its twin

$$e^{-i\theta} = \cos\theta - i \sin\theta \tag{2}$$

giving the result

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \text{and} \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2\, i} \tag{3}$$

Sinθ and cosθ are parameters for the circle; corresponding parameters for the hyperbola are sinhθ and coshθ, defined as

$$\cosh\theta = \frac{e^{\theta} + e^{-\theta}}{2} \quad \text{and} \quad \sinh\theta = \frac{e^{\theta} - e^{-\theta}}{2} \tag{4}$$

We can connect sin and sinh:

$$\sin(i\theta) = \frac{e^{-\theta} - e^{\theta}}{2\, i} = \frac{i^2 \left(e^{\theta} - e^{-\theta}\right)}{2\, i} = \frac{i \left(e^{\theta} - e^{-\theta}\right)}{2} = i \sinh\theta \tag{5}$$

Connecting the standard integrals

Now consider $\int \frac{1}{\sqrt{x^2+1}}\, dx$ which we write as $\int \frac{1}{\sqrt{1+x^2}}\, dx$

Using $1 + x^2 = 1 - i^2 x^2 = 1 - (i\, x)^2$ we have

$$\int \frac{1}{\sqrt{1+x^2}}\, dx = \int \frac{1}{\sqrt{1-(ix)^2}}\, dx = \frac{1}{i} \sin^{-1}(i\, x) \tag{6}$$

Now if $\theta = \frac{1}{i} \sin^{-1}(i\, x)$ then $x = \frac{1}{i} \sin(i\, \theta) = \sinh\theta$ from (5). Hence

$$\int \frac{1}{\sqrt{1+x^2}}\, dx = \sinh^{-1} x \tag{7}$$

Using the definition of sinh θ in (4) we have

$$\sinh\theta = x \rightarrow \frac{e^{\theta} - e^{-\theta}}{2} = x \tag{8}$$

ie

$$e^{\theta} - \frac{1}{e^{\theta}} = 2x \tag{9}$$

so

$$e^{2\theta} - 2x\,e^{\theta} - 1 = 0 \tag{10}$$

The solution to this quadratic is found firstly from

$$\begin{aligned} e^{\theta} &= \frac{2x \pm \sqrt{4x^2 + 4}}{2} \\ &= x \pm \sqrt{x^2 + 1} \end{aligned} \tag{11}$$

Since $e^{\theta} > 0$ we only have one solution, $x + \sqrt{x^2 + 1}$, and then taking logs we have

$$\theta = \text{Log}\left[x + \sqrt{x^2 + 1}\right] \tag{12}$$

Putting it together

We have finally, from (6) and (12)

$$\int \frac{1}{\sqrt{1 + x^2}}\,dx = \frac{1}{i}\sin^{-1}(i\,x) = \theta = \text{Log}\left[x + \sqrt{x^2 + 1}\right] \tag{13}$$

also from (7)

$$\int \frac{1}{\sqrt{1 + x^2}}\,dx = \sinh^{-1} x \tag{14}$$

so

$$\int \frac{1}{\sqrt{1 + x^2}}\,dx = \text{Log}\left[x + \sqrt{x^2 + 1}\right] = \sinh^{-1} x \tag{15}$$