Integration of some relatively simple functions

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This note looks at how primitives of some relatively simple functions are obtained.

(1) $\int \sec\theta \, d\theta = \ln |\sec\theta + \tan\theta|$

When we differentiate $\sec\theta + \tan\theta$ we find

$$\frac{d}{d\theta}(\sec\theta + \tan\theta) = \frac{d}{d\theta}\left(\frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta}\right)$$

$$= \frac{\sin\theta}{\cos^2\theta} + \frac{1}{\cos^2\theta}$$

$$= \frac{1}{\cos\theta}\frac{\sin\theta}{\cos\theta} + \frac{1}{\cos^2\theta} \qquad (1)$$

$$= \sec\theta\tan\theta + \sec^2\theta$$

$$= \sec\theta(\sec\theta + \tan\theta)$$

Hence

$$\sec\theta = \frac{\frac{d}{d\theta}(\sec\theta + \tan\theta)}{\sec\theta + \tan\theta}$$

$$\therefore \int \sec\theta \, d\theta = \int \frac{\frac{d}{d\theta}(\sec\theta + \tan\theta)}{\sec\theta + \tan\theta} \, d\theta \qquad (2)$$

$$= \ln|\sec\theta + \tan\theta|$$

where ln is $\log_e$

(2) $\int \sqrt{x^2 - 1}\, dx = \frac{1}{2} x \sqrt{x^2 - 1} - \frac{1}{2} \ln \left| x + \sqrt{x^2 - 1} \right|$

We can determine the primitive function in a number of ways, two of which are shown below. Both use the hyperbolic functions sinh θ and cosh θ :

$$\sinh\theta = \frac{1}{2}\left(e^{\theta} - e^{-\theta}\right) \qquad \text{oosh}\,\theta = \frac{1}{2}\left(e^{\theta} + e^{-\theta}\right) \tag{3}$$

with properties

$$\begin{aligned} &\cosh^2\theta - \sinh^2\theta = 1 \\ &\cosh 2\theta = \cosh^2\theta + \sinh^2\theta = 1 + 2\sinh^2\theta \\ &\sinh 2\theta = 2\sinh\theta\cosh\theta \\ &\frac{d}{d\theta}(\sinh\ \theta) = \cosh\theta \\ &\frac{d}{d\theta}(\cosh\ \theta) = \sinh\theta \end{aligned} \tag{4}$$

(2a) Substitute $x = \cosh\ \theta$

If $x = \cosh\ \theta$ ($\theta \geq 0$ for the transformation to have an inverse) then $\frac{dx}{d\theta} = \sinh\theta$ and

$$\sqrt{x^2 - 1}\ dx = \sqrt{\cosh^2\theta - 1}\ \sinh\theta\, d\theta = \sinh^2\theta\ d\theta.$$

Hence

$$\begin{aligned} \int \sqrt{x^2 - 1}\ dx &= \int \sinh^2\theta\, d\theta \\ &= \int \frac{1}{2}(\cosh 2\theta - 1)\, d\theta \\ &= \frac{1}{4}\sinh 2\theta - \frac{1}{2}\theta \\ &= \frac{1}{4}(2\sinh\theta\cosh\theta) - \frac{1}{2}\theta \end{aligned} \tag{5}$$

Now $\cosh\ \theta = x$ and $\sinh\ \theta = \sqrt{\cosh^2\theta - 1} = \sqrt{x^2 - 1}$ so $2\sinh\ \theta\cosh\ \theta = 2\,x\sqrt{x^2 - 1}$

Also, we have $x = \cosh\ \theta = \frac{1}{2}\left(e^{\theta} + e^{-\theta}\right)$ and we need to find the inverse function, θ in terms of x.

$$e^{\theta} + e^{-\theta} = 2x \quad \text{so} \quad e^{2\theta} - 2x\,e^{\theta} + 1 = 0 \tag{6}$$

This is a quadratic in e^{θ} and using the formula we have

$$e^{\theta} = \frac{2x + \sqrt{4x^2 - 4}}{2} = x + \sqrt{x^2 - 1} \tag{7}$$

(We only have one value since $x - \sqrt{x^2 - 1}$ would require $e^{\theta} < 1 \Rightarrow \theta < 0$ but we stated above that

$\theta \geq 0$ for the transformation to have an inverse).

Hence

$$\theta = \ln \left| x + \sqrt{x^2 - 1} \right| \tag{8}$$

Finally (5) becomes

$$\int \sqrt{x^2 - 1}\, dx = \frac{1}{4}\left(2x\sqrt{x^2-1}\right) - \frac{1}{2}\ln\left|x + \sqrt{x^2-1}\right|$$
$$= \frac{1}{2}x\sqrt{x^2-1} - \frac{1}{2}\ln\left|x + \sqrt{x^2-1}\right| \tag{9}$$

(2b) Integration by parts and substitute $x = \cosh\theta$

$$\int 1.\sqrt{x^2-1}\, dx = x\sqrt{x^2-1} - \int \frac{x^2}{\sqrt{x^2-1}}\, dx$$
$$= x\sqrt{x^2-1} - \int \frac{x^2-1+1}{\sqrt{x^2-1}}\, dx$$
$$= x\sqrt{x^2-1} - \int \sqrt{x^2-1}\, dx - \int \frac{1}{\sqrt{x^2-1}}\, dx \tag{10}$$

So $$2\int \sqrt{x^2-1}\, dx = x\sqrt{x^2-1} - \int \frac{1}{\sqrt{x^2-1}}\, dx$$

Now to find the integral on the right we use the substitution $x = \cosh\theta$ with $dx = \sinh\theta\, d\theta$:

$$\int \frac{1}{\sqrt{x^2-1}}\, dx = \int \frac{1}{\sqrt{\cosh^2\theta - 1}} \sinh\theta\, d\theta = \int 1\, d\theta = \theta \tag{11}$$

and from (8) we see that $\theta = \ln\left|x + \sqrt{x^2-1}\right|$. Hence we have

$$2\int \sqrt{x^2-1}\, dx = x\sqrt{x^2-1} - \ln\left|x + \sqrt{x^2-1}\right| \tag{12}$$

or

$$\int \sqrt{x^2-1}\, dx = \frac{1}{2}x\sqrt{x^2-1} - \frac{1}{2}\ln\left|x + \sqrt{x^2-1}\right| \tag{13}$$