Means -Arithmetic, Geometric and Harmonic

Dr Richard Kenderdine
Kenderdine Maths Tutoring

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This note looks at three types of Means, the purposes for which they are used and the relationships between them.

Arithmetic Mean

The Arithmetic Mean, A, of two numbers a and b is given by

$$A = \frac{a+b}{2} \tag{1}$$

A is the midpoint of a and b. That is, the difference between A and a is the same as the difference between A and b.

The Arithmetic Mean of n numbers is given by

$$A = \frac{a_1 + a_2 + \ldots\ldots + a_n}{n} \tag{2}$$

So in general we add up the numbers and divide by how many numbers we have. In (1) we said that A is the midpoint of a and b. Another term we can use is *balance point*. This means the sum of the differences between A and all the numbers greater than A equals the sum of the differences between A and all the numbers less than A. For example, suppose we have the numbers 2, 5, 10 and 19 for which $A = 9$. The differences between the numbers and 9 are 7, 4, 1 and 10. The sum of the differences to the numbers less than 9 is $7 + 4 = 11$ and the sum of the differences to the numbers greater than 9 is $1 + 10 = 11$.

The Arithmetic Mean is commonly referred to as the *average* and has many applications eg the average exam mark for a group of students, the average maximum temperature in a calendar month, the average number of calls to a call centre between 8am and 9am.

Geometric Mean

The Geometric Mean, G, of two *positive* numbers a and b is given by

$$G = \sqrt{a\,b} \tag{3}$$

and for n numbers

$$G = \sqrt[n]{a_1\,a_2\,\ldots\ldots\,a_n} \tag{4}$$

The Geometric Mean is used when numbers are multiplied. For example, successive multiplication by 4 and 16 is the same as multiplying by 8 twice because $4 \times 16 = 64 = 8^2$.

A useful application occurs with percentage increases / decreases. For example, an increase of 5% followed by an increase of 10% produces an overall increase of $1.05 \times 1.10 = 1.155$ ie 15.5%. This is the same as two successive equal increases of 7.4709% since $1.074709^2 = 1.155$. Thus the Geometric Mean of 5% and 10% is 7.4709%.

The Geometric Mean is always less than or equal to the Arithmetic Mean. The proof for two positive numbers a and b is:

$$(a-b)^2 \geq 0 \tag{5}$$

$$a^2 - 2\,a\,b + b^2 \geq 0 \tag{6}$$

$$a^2 + 2\,a\,b + b^2 \geq 4\,a\,b \tag{7}$$

$$(a+b)^2 \geq 4\,a\,b \tag{8}$$

$$\left(\frac{a+b}{2}\right)^2 \geq a\,b \tag{9}$$

$$\frac{a+b}{2} \geq \sqrt{a\,b} \tag{10}$$

$$A \geq G \tag{11}$$

with equality when $a = b$.

The proof for $n = 3$ positive numbers requires knowledge of the factorisation [Ref A]:

$$x^3 + y^3 + z^3 - 3\,x\,y\,z = (x+y+z)\left(x^2 + y^2 + z^2 - x\,y - x\,z - y\,z\right) \tag{12}$$

$$= (x+y+z)\,\frac{1}{2}\left(2\,x^2 + 2\,y^2 + 2\,z^2 - 2\,x\,y - 2\,x\,z - 2\,y\,z\right) \tag{13}$$

$$= (x+y+z)\,\frac{1}{2}\left[(x-y)^2 + (x-z)^2 + (y-z)^2\right] \geq 0 \tag{14}$$

Thus $x^3 + y^3 + z^3 - 3\,x\,y\,z \geq 0$, with equality when $x = y = z$.

Now let $a = x^3$, $b = y^3$ and $c = z^3$ giving

$$a + b + c - 3\,\sqrt[3]{a}\,\sqrt[3]{b}\,\sqrt[3]{c} \geq 0 \tag{15}$$

$$\frac{a+b+c}{3} \geq \sqrt[3]{a\,b\,c} \tag{16}$$

$$A \geq G \tag{17}$$

The proof for n numbers is more complicated. One method is to use induction [Ref A].

Harmonic Mean

The Harmonic Mean, H, of two positive numbers a and b is determined from

$$\boldsymbol{\frac{1}{H} = \frac{1}{2}\left(\frac{1}{a} + \frac{1}{b}\right)} \tag{18}$$

$$= \frac{1}{2}\left(\frac{a+b}{a\,b}\right) \tag{19}$$

That is, the reciprocal of the Harmonic Mean is the Arithmetic Mean of the reciprocals of the numbers.

Alternatively,

$$\boldsymbol{H = \frac{2\,a\,b}{a+b}} \tag{20}$$

The Harmonic Mean is used with inverse relationships. For example, speed and time are inversely related: for a fixed distance, increasing the speed results in a quicker journey time and vice versa. Suppose we have an out and back journey of 100km each way with the speed 25 kph out and 50 kph back (think peak hour / non-peak hour, a cyclist cycling into wind and then with the wind, a vessel sailing against the current then with the current). The outward journey takes $\frac{100}{25} = 4$ hours and the return only $\frac{100}{50} = 2$ hours. The total distance is 200km in 6 hours, giving an average speed of $\frac{200}{6} = 33.\dot{3}$ kph. This is the Harmonic Mean and can be calculted from (20) as $\frac{2\times25\times50}{25+50} = \frac{2500}{75} = 33.3$ kph. (It is necessary for the numerators in the inverse relationship, here 100km, to be the same).

For n positive numbers we have

$$\boldsymbol{\frac{1}{H} = \frac{1}{n}\left[\frac{1}{a_1} + \frac{1}{a_2} + \ldots.. + \frac{1}{a_n}\right]} \tag{21}$$

The Harmonic Mean is less than or equal to the Geometric Mean (again, equality occurs when all the numbers are equal).

We have, for two numbers,

$$H = \frac{2}{a+b}(a\,b) \tag{22}$$

$$= \frac{1}{A} \times G^2 \tag{23}$$

$$\leq \frac{1}{G} \times G^2 = G \tag{24}$$

We have the inequality in (24) because $A \geq \mathrm{G} \Rightarrow \frac{1}{A} \leq \frac{1}{G}$

For n positive numbers, since we know the Arithmetic Mean is greater than or equal the Geometric Mean [Ref A],

$$\frac{1}{H} = \frac{1}{n}\left[\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}\right] \geq \sqrt[n]{\frac{1}{a_1}\,\frac{1}{a_2}\,\ldots\,..\,\frac{1}{a_n}} \tag{25}$$

$$\sqrt[n]{a_1\, a_2\, \ldots\, ..\, a_n} \geq \frac{1}{\frac{1}{n}\left[\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}\right]} \tag{26}$$

$$G \geq \frac{1}{\frac{1}{H}} = H \tag{27}$$

Thus we have

$$\boldsymbol{H \leq G \leq A} \tag{28}$$

Finally, from (23) we see that

$$\boldsymbol{G^2 = A\,H \;\Rightarrow\; G = \sqrt{A\,H}\,.} \tag{29}$$

That is, the Geometric Mean of two numbers is the Geometric Mean of the Arithmetic and Harmonic Means of the two numbers.

Here we calculate the three means of the numbers 2 and 23, together with the Geometric Mean of the Harmonic and Arithmetic Means:

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In[6]:= data = {2, 23}; {HarmonicMean[data] // N, GeometricMean[data] // N,
          Mean[data] // N, GeometricMean[{HarmonicMean[data], Mean[data]}] // N}

Out[6]= {3.68, 6.78233, 12.5, 6.78233}
```

We see that $G = \sqrt{A\,H}$ is true. However it is not true in general, as shown with five numbers:

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In[4]:= data = {2, 14, 23, 7, 19};
        {HarmonicMean[data] // N, GeometricMean[data] // N, Mean[data],
         GeometricMean[{HarmonicMean[data], Mean[data]}] // N}

Out[4]= {6.16983, 9.69499, 13, 8.95588}
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References

A. *The USSR Olympiad Problem Book*, D.O. Shklarsky, N.N. Chentzov, I.M. Yaglom, Dover Publications, New York, 1993

B. *CRC Concise Encyclopedia of Mathematics,* Eric W. Weisstein, Chapman and Hall/CRC, 2003