Sums of powers of the natural numbers

Dr Richard Kenderdine
Kenderdine Maths Tutoring

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A common question asked of students of mathematical induction is to prove $1^2 + 2^2 + \ldots\ldots + n^2 = \frac{1}{6}\, n(2\,n + 1)\,(n + 1)$. This note derives the expression on the right-hand side as well as expressions for higher powers.

Sum of the first *n* natural numbers

The series 1 + 2 + 3 +.....+ n is an Arithmetic Progression with first term = 1, difference = 1 and n terms. The standard formula for such a series is $\frac{n}{2}\,(a + l) = \frac{1}{2}\, n(1 + n)$ which we can also write as $\frac{1}{2}\, n^2 + \frac{1}{2}\, n$.

Thus we have

$$\sum_{k=1}^{n} k = \frac{n}{2}\,(1 + n) = \frac{1}{2}\, n^2 + \frac{1}{2}\, n \tag{1}$$

The telescopic property

The telescopic property is useful for deriving results relating to series. Consider

$$\sum_{k=1}^{n} [a(k + 1) - a(k)] \;=\; [a(2) - a(1)] + [a(3) - a(2)] + [a(4) - a(3)] + \ldots.[a(n) - a(n - 1)] + [a(n + 1) - a(n)] \tag{2}$$

$$= a(n + 1) - a(1) \tag{3}$$

Sum of the first *n* square numbers

Consider

$$\sum_{k=1}^{n} \left[(k + 1)^3 - k^3\right] \tag{4}$$

Using the telescopic property this becomes

$$\sum_{k=1}^{n} \left[(k + 1)^3 - k^3\right] = (n + 1)^3 - 1^3 \tag{5}$$

$$= \; n^3 + 3\,n^2 + 3\,n \tag{6}$$

Alternatively we can expand out the summation

$$\sum_{k=1}^{n} \left[(k + 1)^3 - k^3\right] = \sum_{k=1}^{n} \left[k^3 + 3\,k^2 + 3\,k + 1 \;-\; k^3\right] \tag{7}$$

$$= \; 3\sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1 \tag{8}$$

$$= 3\sum_{k=1}^{n} k^2 + 3\left[\frac{n}{2}\,(1 + n)\right] + n \tag{9}$$

$$= 3\sum_{k=1}^{n} k^2 + \frac{3}{2}\, n^2 + \frac{5}{2}\, n \tag{10}$$

Equating (6) with (10),

$$3\sum_{k=1}^{n} k^2 + \frac{3}{2}\, n^2 + \frac{5}{2}\, n = n^3 + 3\,n^2 + 3\,n \tag{11}$$

So

$$\sum_{k=1}^{n} k^2 = \frac{1}{3}\left(n^3 + \frac{3}{2} n^2 + \frac{1}{2} n\right) \tag{12}$$

$$= \frac{1}{3} n^3 + \frac{1}{2} n^2 + \frac{1}{6} n \tag{13}$$

or in factored form

$$\sum_{k=1}^{n} k^2 = \frac{1}{6} n(2\,n + 1)\,(n + 1) \tag{14}$$

Sum of the first *n* cubed numbers

Consider

$$\sum_{k=1}^{n} \left[(k + 1)^4 - k^4\right] \tag{15}$$

Using the telescopic property this becomes

$$\sum_{k=1}^{n} \left[(k + 1)^4 - k^4\right] = (n + 1)^4 - 1^4 \tag{16}$$

$$= n^4 + 4\,n^3 + 6\,n^2 + 4\,n \tag{17}$$

Alternatively we can expand out the summation

$$\sum_{k=1}^{n} \left[(k + 1)^4 - k^4\right] = \sum_{k=1}^{n} \left[k^4 + 4\,k^3 + 6\,k^2 + 4\,k + 1 - k^4\right] \tag{18}$$

$$= 4 \sum_{k=1}^{n} k^3 + 6 \sum_{k=1}^{n} k^2 + 4 \sum_{k=1}^{n} k + \sum_{k=1}^{n} 1 \tag{19}$$

$$= 4 \sum_{k=1}^{n} k^3 + n(2\,n + 1)\,(n + 1) + 4\left[\frac{n}{2}\,(1 + n)\right] + n \tag{20}$$

$$= 4 \sum_{k=1}^{n} k^3 + 2\,n^3 + 5\,n^2 + 4\,n \tag{21}$$

Equating (17) with (21),

$$4 \sum_{k=1}^{n} k^3 + 2\,n^3 + 5\,n^2 + 4\,n = n^4 + 4\,n^3 + 6\,n^2 + 4\,n \tag{22}$$

So

$$\sum_{k=1}^{n} k^3 = \frac{1}{4}\left(n^4 + 2\,n^3 + n^2\right) \tag{23}$$

$$= \frac{1}{4} n^4 + \frac{1}{2} n^3 + \frac{1}{4} n^2 \tag{24}$$

or in factored form

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n + 1)}{2}\right)^2 \tag{25}$$

Note that since $\frac{n(n+1)}{2}$ is the sum of the first n natural numbers then we have

$$\mathbf{1^3 + 2^3 + 3^3 + \ldots \ldots + n^3 = (1 + 2 + 3 + \ldots \ldots + n)^2} \tag{26}$$

Sum of the first n quartic numbers

Proceeding as previously

$$\sum_{k=1}^{n}\left[(k+1)^5-k^5\right]=(n+1)^5-1^5 \tag{27}$$

which becomes

$$5\sum_{k=1}^{n}k^4+10\sum_{k=1}^{n}k^3+10\sum_{k=1}^{n}k^2+5\sum_{k=1}^{n}k+\sum_{k=1}^{n}1=n^5+5\,n^4+10\,n^3+10\,n^2+5\,n \tag{28}$$

and using substitutions for $\sum_{k=1}^{n}k^3$, $\sum_{k=1}^{n}k^2$ and $\sum_{k=1}^{n}k$ from above (28) simplifies to

$$\sum_{k=1}^{n}k^4=\frac{1}{5}n^5+\frac{1}{2}n^4+\frac{1}{3}n^3-\frac{1}{30}n \tag{29}$$

or in factored form

$$\sum_{k=1}^{n}k^4=\frac{1}{30}(2\,n+1)\,n(n+1)[3\,n(n+1)-1] \tag{30}$$

Summary so far

The sums of the first four powers of the natural numbers are:

$$\sum_{k=1}^{n}k=\frac{1}{2}n^2+\frac{1}{2}n.$$

$$\sum_{k=1}^{n}k^2=\frac{1}{3}n^3+\frac{1}{2}n^2+\frac{1}{6}n$$

$$\sum_{k=1}^{n}k^3=\frac{1}{4}n^4+\frac{1}{2}n^3+\frac{1}{4}n^2$$

$$\sum_{k=1}^{n}k^4=\frac{1}{5}n^5+\frac{1}{2}n^4+\frac{1}{3}n^3-\frac{1}{30}n$$

Considering the general power p, the similarities of the expressions for $\sum_{k=1}^{n}k^p$ include

- the first term is $\int n^p\,dn$
- the second coefficient is $\frac{1}{2}$
- powers of n decrease from p+1 to 1 (the coefficients for n when $p=3$ and n^2 when $p=4$ are 0)
- the coefficients sum to 1 in each case

In factored form we have

$$\sum_{k=1}^{n}k=\frac{1}{2}n(n+1).$$

$$\sum_{k=1}^{n}k^2=\frac{1}{6}(2\,n+1)\,n(n+1)$$

$$\sum_{k=1}^{n}k^3=\frac{1}{4}[n\,(n+1)]^2$$

$$\sum_{k=1}^{n}k^4=\frac{1}{30}(2\,n+1)\,n(n+1)[3\,n(n+1)-1]$$

It can be seen that the RHS is a polynomial in $n(n+1)$ with the extra factor $2n+1$ when p is even.

Historical note

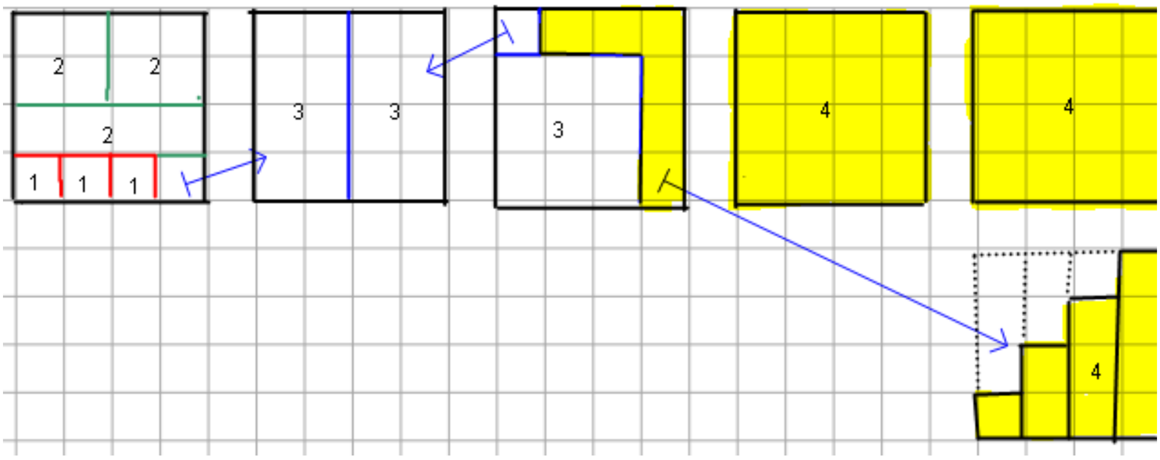
Archimedes (287-212 BC) stated and proved a formula for the sum of squares as Proposition 10 in his treatise translated as *On Spirals*:

If a series of any number of lines be given, which exceed one another by an equal amount; and the difference be equal to the least, and if other lines be given equal in number to these and in quantity to the greatest, the squares on the lines equal to the greatest, plus the square on the greatest and the rectangle contained by the least and the sum of all those exceeding one another by an equal amount will be the triplicate of all the squares on the lines exceeding one another by an equal amount.

In modern terms, Archimedes is referring to an arithmetic progression with difference equal to the first term. If we make both equal to 1 then the proposition states:

$$n^2(n+1) + (1+2+3+\ldots.+n) = 3\left(1^2+2^2+3^2+\ldots.+n^2\right) \tag{31}$$

This is illustrated below for $n = 4$ where there are 5 lots of 4^2 plus $(1+2+3\ +4)$ cut up into 3 lots each of 1^2, 2^2, 3^2 and 4^2:



Now using $(1+2+3+\ldots+n) = \frac{n}{2}(1+n)$ we have

$$n^2(n+1) + \frac{n}{2}(n+1) = 3\left(1^2+2^2+3^2+\ldots.+n^2\right) \tag{32}$$

$$(n+1)\left(n^2+\frac{n}{2}\right) = 3\left(1^2+2^2+3^2+\ldots.+n^2\right) \tag{33}$$

$$\frac{1}{2}(n+1)\,n(2\,n+1) = 3\left(1^2+2^2+3^2+\ldots.+n^2\right) \tag{34}$$

$$\frac{1}{6}(2\,n+1)\,n(n+1) = 1^2+2^2+3^2+\ldots.+n^2 \tag{35}$$

as previously derived.

Besides the ancient Greeks, Hindus and Arabs had rules for summing powers up to and including 4. In 1631 Johann Faulhaber (1580 - 1635) published *Academiae Algebrae* that contained the sums up to $p = 17$. However there was no general formula for arbitrary p until Jacques (some references use Jakob or Jacob) Bernoulli (1654 - 1705) provided a solution that was not published until 1713 in *Ars Conjectandi.* Bernoulli listed the results up to power 10 and described the general pattern, without proof. The pattern required the use of what came to be termed `Bernoulli numbers` that have no obvious pattern but can be generated in a number of ways.

Bernoulli numbers

One way of obtaining these numbers is from the coefficients in the power series expansion of

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n\, x^n}{n!} \tag{36}$$

Mathematica gives the expansion, up to x^{12}, as

```
Series[x/(Exp[x] - 1), {x, 0, 12}]
```

$$1-\frac{x}{2}+\frac{x^2}{12}-\frac{x^4}{720}+\frac{x^6}{30\,240}-\frac{x^8}{1\,209\,600}+\frac{x^{10}}{47\,900\,160}-\frac{691\,x^{12}}{1\,307\,674\,368\,000}+O[x]^{13}$$

which can be written as

$$1-\frac{1}{2}\frac{x}{1!}+\frac{1}{6}\frac{x^2}{2!}-\frac{1}{30}\frac{x^4}{4!}+\frac{1}{42}\frac{x^6}{6!}\ldots\ldots\ldots \tag{37}$$

Thus the Bernoulli numbers are (B_0 =1, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_3 = 0$, $B_4 = -\frac{1}{30}$, $B_5 = 0$, $B_6 = \frac{1}{42}$, $B_7 = 0$, ...) . Note that $B_n = 0$ for n odd except $n = 1$.

Another way is the use of Bernoulli polynomials, $B_n(x)$, which appear in the expansion

$$\frac{t\,e^{x\,t}}{e^t-1}=\sum_{n=0}^{\infty}\frac{B_n(x)\,t^n}{n!} \tag{38}$$

Mathematica gives the expansion, up to t^6, as

```
Series[t e^(x t)/(e^t - 1), {t, 0, 6}]
```

$$1+\left(-\frac{1}{2}+x\right)t+\frac{1}{12}\left(1-6x+6x^2\right)t^2+\frac{1}{12}\left(x-3x^2+2x^3\right)t^3+\frac{1}{720}\left(-1+30x^2-60x^3+30x^4\right)t^4+\frac{1}{720}\left(-x+10x^3-15x^4+6x^5\right)t^5+\frac{\left(1-21x^2+105x^4-126x^5+42x^6\right)t^6}{30\,240}+O[t]^7$$

This expansion can be re-written as

$$1\frac{t^0}{0!}+\left(x-\frac{1}{2}\right)\frac{t^1}{1!}+\left(x^2-x+\frac{1}{6}\right)\frac{t^2}{2!}+\left(x^2-\frac{3}{2}x+\frac{1}{2}x\right)\frac{t^3}{3!}+\left(x^4-2x^3+x^2-\frac{1}{30}\right)\frac{t^4}{4!}\ldots\ldots \tag{39}$$

The Bernoulli polynomials are the coefficients of $\frac{t^n}{n!}$ and the Bernoulli numbers are the constant terms of the polynomials.

One property of the Bernoulli numbers is

$$\binom{k+1}{1}B_k+\binom{k+1}{2}B_{k-1}+\binom{k+1}{3}B_{k-2}+\ldots\ldots+\binom{k+1}{k}B_0=0 \tag{40}$$

The general formula using Bernoulli numbers

In modern notation we can write

$$\sum_{k=1}^{n}k=\frac{1}{2}\left[\binom{2}{0}n^2+\binom{2}{1}\frac{1}{2}n\right] \tag{41}$$

$$\sum_{k=1}^{n}k^2=\frac{1}{3}\left[\binom{3}{0}n^3+\binom{3}{1}\frac{1}{2}n^2+\binom{3}{2}B_2\,n\right] \tag{42}$$

$$\sum_{k=1}^{n}k^3=\frac{1}{4}\left[\binom{4}{0}n^4+\binom{4}{1}\frac{1}{2}n^3+\binom{4}{2}B_2\,n^2+\binom{4}{3}B_3\,n\right] \tag{43}$$

Note that in (43) $B_3 = 0$.

The pattern can be summarised as

$$\sum_{k=1}^{n}k^p=\frac{1}{p+1}\left[\binom{p+1}{0}n^{p+1}+\binom{p+1}{1}\frac{1}{2}n^p+\sum_{j=2}^{p}\binom{p+1}{j}B_j\,n^{p+1-j}\right] \tag{44}$$

Let's test this by summing powers of the first 10 natural numbers. First set up a function for arbitrary n and p.

In[14]:=
```
sumpower[n_, p_] :=
  1/(p + 1) (Binomial[p + 1, 0] n^(p+1) + 1/2 Binomial[p + 1, 1] n^p +
    Sum[Binomial[p + 1, j] BernoulliB[j] n^(p+1-j), {j, 2, p}]);
```

Now sum powers of the first 10 natural numbers, from power 1 to power 10:

In[16]:=
```
Table[sumpower[10, k], {k, 1, 10}]
```

Out[16]=
```
{55, 385, 3025, 25 333, 220 825, 1 978 405,
 18 080 425, 167 731 333, 1 574 304 985, 14 914 341 925}
```

For example $1^2 + 2^2 + 3^2 + \ldots + 10^2 = 385$ and $1^3 + 2^3 + 3^3 + \ldots + 10^3 = 3025$.

Now use the summation $\sum_{k=1}^{10} k^p$ for $1 \leq p \leq 10$

In[17]:=
```
Table[Sum[j^p, {j, 10}], {p, 1, 10}]
```

Out[17]=
```
{55, 385, 3025, 25 333, 220 825, 1 978 405,
 18 080 425, 167 731 333, 1 574 304 985, 14 914 341 925}
```

Thus we have agreement.

References

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