Location of Inflection Points relative to Stationary Points

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Observant students sometimes notice that the x-coordinate of an inflection point lies midway between the x-coordinates of two stationary points. Does this always occur? Lets look at some polynomials.

Cubic polynomials

In order to have two stationary points the polynomial must be at least of degree 3. Cubics can have 0, 1 or 2 stationary points and we are concerned here with the latter case.

Let $p(x) = ax^3 + bx^2 + cx + d$ have stationary points at $x = \alpha$ and $x = \beta$.

Hence $p'(x) = 3ax^2 + 2bx + c = 0$ has solutions $x = \alpha, \beta$.

(Note: a cubic will have two stationary points if the discriminant of the derivative is positive. This results in the condition $b^2 - 3ac > 0$).

The sum of these roots is $\alpha + \beta = -\frac{2b}{3a}$

Now the inflection point occurs when $p''(x) = 0$. Here we have $p''(x) = 6ax + 2b = 0$ when $x = -\frac{2b}{6a} = -\frac{b}{3a}$

Since $\alpha + \beta = -\frac{2b}{3a}$ then $-\frac{b}{3a} = \frac{\alpha+\beta}{2}$

ie the x-coordinate of the inflection point is the average of the x-coordinates of the stationary points.

This result can also be determined by reasoning. The derivative of a cubic is a quadratic which must have two x-intercepts if there are two stationary points. The inflection point of the cubic occurs at the turning point of the quadratic and this occurs at the axis of symmetry of the quadratic ie at the average of the x-coordinates of the stationary points.

Note that the stationary points will be turning points because $p''(x)$ is linear and hence will have one root ie there is only one inflection point.

Now consider the y-coordinates.

The y-coordinate at $x = \alpha$ and β are respectively $a\alpha^3 + b\alpha^2 + c\alpha + d$ and $a\beta^3 + b\beta^2 + c\beta + d$ with an average value

$$a\left(\frac{\alpha^3+\beta^3}{2}\right) + b\left(\frac{\alpha^2+\beta^2}{2}\right) + c\left(\frac{\alpha+\beta}{2}\right) + d \tag{1}$$

The y-coordinate when $x = \frac{\alpha+\beta}{2}$, ie the inflection point, is given by

$$a\left(\frac{\alpha+\beta}{2}\right)^3 + b\left(\frac{\alpha+\beta}{2}\right)^2 + c\left(\frac{\alpha+\beta}{2}\right) + d \tag{2}$$

Comparing (1) and (2) we see that the last two terms are identical. Now we need to find the relationship between the first two terms. Consider (1) first:

$$a\left(\frac{\alpha^3+\beta^3}{2}\right) + b\left(\frac{\alpha^2+\beta^2}{2}\right) = \frac{a}{2}\left(\alpha^3+\beta^3\right) + \frac{b}{2}\left(\alpha^2+\beta^2\right) \tag{3}$$

$$= \frac{a}{2}(\alpha+\beta)\left(\alpha^2-\alpha\beta+\beta^2\right)+\frac{b}{2}\left(\alpha^2+\beta^2\right) \tag{4}$$

$$= \left(\alpha^2+\beta^2\right)\left(\frac{a}{2}(\alpha+\beta)+\frac{b}{2}\right)-\frac{a}{2}(\alpha+\beta)(\alpha\beta) \tag{5}$$

$$= \left(\alpha^2+\beta^2\right)\left(\frac{a}{2}\left(-\frac{2\,b}{3\,a}\right)+\frac{b}{2}\right)-\frac{a}{2}\left(-\frac{2\,b}{3\,a}\right)(\alpha\beta) \tag{6}$$

$$= \left(\alpha^2+\beta^2\right)\left(-\frac{b}{3}+\frac{b}{2}\right)+\frac{b}{3}(\alpha\beta) \tag{7}$$

$$= \frac{b}{6}\left(\alpha^2+\beta^2\right)+\frac{b}{3}(\alpha\beta) \tag{8}$$

$$= \frac{b}{6}\left(\alpha^2+2\,\alpha\beta+\beta^2\right) \tag{9}$$

$$= \frac{b}{6}(\alpha+\beta)^2 \tag{10}$$

Note in (6) the substitution $\alpha+\beta=-\frac{2\,b}{3\,a}$ as shown previously.

Now consider (2):

$$a\left(\frac{\alpha+\beta}{2}\right)^3+b\left(\frac{\alpha+\beta}{2}\right)^2=\frac{a}{8}(\alpha+\beta)^3+\frac{b}{4}(\alpha+\beta)^2 \tag{11}$$

$$= (\alpha+\beta)^2\left(\frac{a}{8}(\alpha+\beta)+\frac{b}{4}\right) \tag{12}$$

$$= (\alpha+\beta)^2\left(\frac{a}{8}\left(-\frac{2\,b}{3\,a}\right)+\frac{b}{4}\right) \tag{13}$$

$$= (\alpha+\beta)^2\left(-\frac{b}{12}+\frac{b}{4}\right) \tag{14}$$

$$= \frac{b}{6}(\alpha+\beta)^2 \tag{15}$$

Hence (10) and (15) are identical. Thus expressions (1) and (2) are equal and so we have the result:

In a cubic with two turning points the inflection point is the midpoint of the turning points.

The coordinates of the inflection point are then

$$\left(\frac{\alpha+\beta}{2},\ \frac{b}{6}(\alpha+\beta)^2+c\left(\frac{\alpha+\beta}{2}\right)+d\right) \tag{16}$$

$$\text{or}\quad\left(\frac{1}{2}(\alpha+\beta),\ \frac{b}{6}(\alpha+\beta)^2+\frac{c}{2}(\alpha+\beta)+d\right) \tag{17}$$

Some interesting points arising from this analysis:

- the form of the y-coordinate in (17) is a quadratic in terms of the sum of the x-coordinates of the turning points
- the inflection point and turning points are collinear
- the plot of the cubic will have point symmetry about the inflection point. This means that if we transform the x and y coordinates such that the origin is at the inflection point, the form of the function will be odd. For example, consider $y=x^3-6\,x^2-15\,x+1$ which has TPs at (-1, 9) and (5, -99) and IP at (2, -45). If we use the transformation $u=x-2$, $v=y+45$ then substitute $x=u+2$, $y=v-45$ we have $v=u^3-27\,u$. Thus the original function, which is neither odd nor even, has been transformed into an odd function and such functions always have point symmetry about the origin.

Quartic polynomials

Suppose the quartic has three stationary points at $x=\alpha,\beta$ and γ and let the derivative of the quartic be $p'(x)=(x-\alpha)(x-\beta)(x-\gamma)$.

The x-coordinates midway between the stationary points will be at $\frac{\alpha+\beta}{2}$ and $\frac{\beta+\gamma}{2}$

Again, the inflection points occur when $p''(x) = 0$.

Here $p''(x) = x^3 - (\alpha + \beta + \gamma)\, x^2 + (\alpha\beta + \alpha\gamma + \beta\gamma)\, x - \alpha\beta\gamma$

The sum of the roots of $p''(x) = 0$ is $\alpha + \beta + \gamma$

The sum of the x-coordinates midway between the stationary points is $\frac{\alpha+\beta}{2} + \frac{\beta+\gamma}{2} = \frac{1}{2}\alpha + \beta + \frac{1}{2}\gamma$

and hence the x-coordinates midway between the stationary points cannot be solutions of $p''(x) = 0$

ie in a quartic, the *x*-coordinates of the inflection point ARE NOT the average of the *x*-coordinates of the stationary points.

Higher-degree polynomials

Similar results occur for higher-degree polynomials.

Consider a quintic with derivative $p'(x) = (x + 3)(x + 1)(x - 2)(x - 8)$

The solutions to $p''(x) = 0$ are $x = -2.16466, 0.65321, 6.01145$ and these are not the average of the stationary point x-coordinates at $x = -3, -1, 2, 8$.

Conclusion

The only polynomial that has the coordinates of the inflection point as the midpoint of the coordinates of the stationary points is the cubic.

Acknowledgment

This note was inspired by a student with a curious nature who liked to look for shortcuts.