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SOME QUESTIONS TO CONSIDER

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1. Find three interesting facts about the decimal forms of $\frac{1}{7}, \frac{2}{7}, ..., \frac{6}{7}$

2. Find the next three numbers in the series $(21, 220, 221, 23, 264, 265)$

3. Inscribe a regular octagon within any given circle

4. Prove $n! \leq (\frac{n+1}{2})^n$ for integers $n \geq 1$

5. Derive an expression in terms of n only for $\sum_{k=1}^{n} k^3$

6. Solve $x^2 - 41y^2 = 1$ for integers x, y

These questions have been chosen to encourage you to go beyond the normal school curriculum. They either require greater understanding of topics already met or exploration of new areas of mathematics. Maybe they will stimulate further research using the internet; for example, some terms that may be unfamiliar are in italics in the answers.

ANSWERS:

1. The decimal equivalents are recurring with period 6. The decimal for $\frac{1}{7}$ is 0.142857 while the others are, in order, (0.285714, 0.428571, 0.571428, 0.714285, 0.857142).
   It can be noted that

   - the same six digits occur in all fractions with the digits 3, 6 and 9 excluded
   - taking the digits 142857 in pairs we note that $2 \times 14 = 28$ and $2 \times 28 = 56$, one less than 57
   - 0.285714 is 0.142857 with the digits 14 cycled to the end and then the 4 is bought back to the start in 0.428571
   - in each case, splitting the six digits into two groups of three and adding results in 9 in each column eg $142857 \Rightarrow 142, 857 \Rightarrow 1+8, 4+5, 2+7$
   - this means that the groups of three are swapped between fractions that add to 1 eg $\frac{3}{7} = 0.428571$, $\frac{4}{7} = 0.571428$ where we see that the last three digits of $\frac{3}{7}$ become the first three digits of $\frac{4}{7}$.

   This is an example of a *cyclic number*. A cyclic number is a number having n-1 digits and when multiplied by $1, 2, ...n$-1 yields the same digits in a different order. Such numbers are generated by $\frac{1}{n}$ where n is 7,17,19,23,29,47,59,61,97,....It is thought that there are an infinite number of cyclic numbers.

   There are results in *Number Theory* that determine the number of recurring digits (the period) in the decimal form of a fraction: the period is the least number x such that 10^x leaves a remainder of 1 when divided by the denominator. Here, if you divide 7 into $(10^1, 10^2, 10^3, 10^4, 10^5, 10^6)$ a remainder of 1 occurs for the first time with 10^6. Hence the period for $\frac{1}{7}$ is 6.

2. There is an easy relationship between primitive Pythagorean triads when the smallest number is odd. (A primitive triad has no common factor between the three numbers eg (3,4,5)).

   The first three examples are (3,4,5), (5,12,13), (7,24,25). Note that the last two numbers always differ by 1 and their sum equals the square of the smallest number. Hence, if n is the smallest number, the other numbers are $(\frac{n^2-1}{2}, \frac{n^2+1}{2})$

   Hence the required numbers are $(25, 312, 313)$ since $25^2 = 625 = 312 + 313$.

3. The steps are

   a) Draw two chords at an angle greater than 45° to each other

   b) Construct the perpendicular bisectors of the chords (compass point on one end of the chord and draw arc, repeat at other end of chord, join intersection points of arcs)

   c) The intersection of the perpendicular bisectors is the centre of the circle

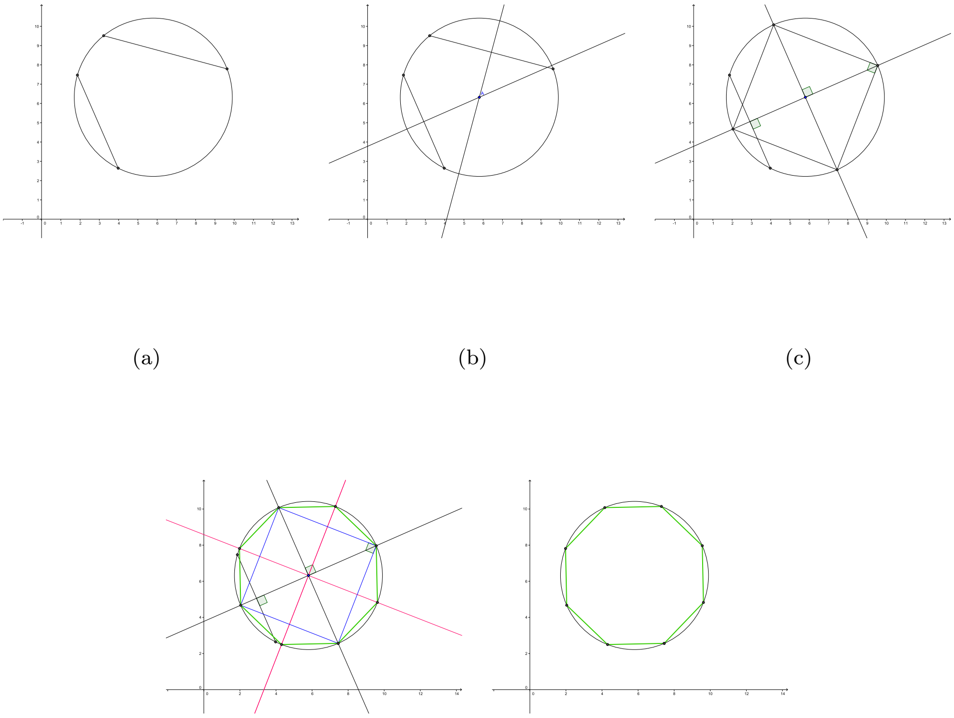
   d) Use one of the perpendicular bisectors as a diameter and then construct the perpendicular bisector of this diameter

   e) Join the end points of the diameters to form a square (the diameters are the diagonals of the quadrilateral and they are equal and bisect each other at 90° therefore it must be a square)

   f) Now construct the perpendicular bisectors of the sides of the square (or bisect the angles at the centre between the diagonals) and intersect them with the circle

   g) Join these intersection points with the vertices of the aquare

   Extension: how many regular polygons can you inscribe in a circle?



(a) (b) (c)

(d) (e)

Figure 1: (a) Chords (b) Finding the centre (c) Construct the square (d) The octagon (e) The end result

4. This is just the relationship between the geometric mean (GM) and the arithmetic mean (AM) for positive numbers.

   Using $(x-y)^2 + 4xy = (x+y)^2$ we see that $4xy \leq (x+y)^2 \Rightarrow \sqrt{xy} \leq \frac{x+y}{2}$

   Now $\sqrt{xy}$ is the GM of x, y and $\frac{x+y}{2}$ is the AM so GM $\leq$ AM for two numbers. The same relationship holds for n numbers.

   The AM of the numbers $(1, 2, 3, \ldots.., n)$ is $\frac{n+1}{2}$ (the average of a sequence of numbers in arithmetic progression is the average of the first and last number in order)

   and the GM is $\sqrt[n]{(1)(2)(3)\ldots.(n-1)n} = \sqrt[n]{n!}$

   Hence $\sqrt[n]{n!} \leq \frac{n+1}{2}$ so $n! \leq (\frac{n+1}{2})^n$

5. There is a formula for such power sums in terms of *Bernoulli Numbers* (Faulhaber's Formula) but an expression can be easily derived for low powers in the following manner.

$$\begin{aligned}\sum_{k=1}^{n} k^4 - \sum_{k=1}^{n}(k-1)^4 &= \sum_{k=1}^{n}[k^4 - (k^4 - 4k^3 + 6k^2 - 4k + 1)] \\ &= \sum_{k=1}^{n} 4k^3 - 6k^2 + 4k - 1 \\ &= \sum_{k=1}^{n}(4k^3) - \sum_{k=1}^{n}(6k^2) + \sum_{k=1}^{n}(4k-1) \\ &= 4\sum_{k=1}^{n} k^3 - 6\sum_{k=1}^{n} k^2 + \sum_{k=1}^{n}(4k-1)\end{aligned} \tag{1}$$

Now LHS $= n^4$ and $4k-1$ is an AP with

$$\sum_{k=1}^{n}(4k-1) = \frac{n}{2}(4n-1+3) = n(2n+1) \tag{2}$$

To obtain $\sum_{k=1}^{n} k^2$ we use a similar approach:

$$\begin{aligned}\sum_{k=1}^{n} k^3 - \sum_{k=1}^{n}(k-1)^3 &= \sum_{k=1}^{n}[k^3 - (k^3 - 3k^2 + 3k - 1)] \\ &= \sum_{k=1}^{n} 3k^2 - (3k-1) \\ &= 3\sum_{k=1}^{n} k^2 - \frac{n}{2}(3n+1)\end{aligned} \tag{3}$$

giving, since LHS$=n^3$,

$$\sum_{k=1}^{n} k^2 = \frac{1}{3}[n^3 + \frac{n}{2}(3n+1)] = \frac{n}{6}(2n+1)(n+1) \tag{4}$$

(This formula is often given as an example to prove by induction)

Then

$$\begin{aligned}\sum_{k=1}^{n} k^3 &= \frac{1}{4}[n^4 + n(2n+1)(n+1) - n(2n+1)] \\ &= \frac{n^4 + 2n^3 + n^2}{4} \\ &= \left[\frac{n(n+1)}{2}\right]^2\end{aligned} \tag{5}$$

It is noted that

$$\sum_{k=1}^{n} k^3 = \left[\sum_{k=1}^{n} k\right]^2 \tag{6}$$

6. This is an example of a *Pell Equation* and the solution is obtained from the *convergents* of the *continued fraction* of $\sqrt{41}$. The recursive formulae for obtaining the continued fraction of an irrational number D are

   - Set $\alpha_0 = D$
   - $a_k = \lfloor \alpha_0 \rfloor$
   - $\alpha_{k+1} = \frac{1}{(\alpha_k - a_k)}$

   where $\lfloor x \rfloor$ means the integer part of x eg $\lfloor 2.75 \rfloor = 2$

   Then the continued fraction is $(a_0; a_1, a_2, ...)$ and for $\sqrt{41}$
   we have $(6; 2, 2, 12, 2, 2, 12, ..)$

   The convergents are given by $\frac{p_k}{q_k}$ where

   $$p_k = a_k p_{k-1} + p_{k-2} \tag{7}$$

   and

   $$q_k = a_k q_{k-1} + q_{k-2} \tag{8}$$

   with $p_0 = a_0,\ q_0 = 1,\ p_1 = a_0 a_1 + 1,\ q_1 = a_1$.

   The continued fraction became periodic at $a_{r+1} = a_3$ giving $r = 2$ and the theory says that the values of x and y are the $k = 2r + 1$ values of (p_k, q_k).

   Working this out gives $(p_5, q_5) = (2049, 320)$. Check $2049^2 - 41 \times 320^2 = 1$.

   The values of (x, y) for the genaral equation $x^2 - Dy^2 = 1$ differ greatly as D varies. For example when $D = 40$ the solution is $(19, 3)$ while $(13, 2)$ satisfies the equation with $D = 42$.